

# THE QUATERNARY COMPLEX HADAMARD MATRICES OF ORDERS 10, 12, AND 14

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**ABSTRACT.** A complete classification of quaternary complex Hadamard matrices of orders 10, 12 and 14 is given, and a new parametrization scheme for obtaining new examples of affine parametric families of complex Hadamard matrices is provided. On the one hand, it is proven that all  $10 \times 10$  and  $12 \times 12$  quaternary complex Hadamard matrices belong to some parametric family, but on the other hand, it is shown by exhibiting an isolated  $14 \times 14$  matrix that there cannot be a general method for introducing parameters into these types of matrices.

**2010 Mathematics Subject Classification.** Primary 05B20, secondary 15B34.

**Keywords and phrases.** *Butson-type Hadamard matrix, Classification, Complex Hadamard matrix.*

## 1. INTRODUCTION

A complex Hadamard matrix  $H$  of order  $n$  is an  $n \times n$  matrix with unimodular entries satisfying  $HH^* = nI$  where  $*$  is the conjugate transpose,  $I$  is the identity matrix of order  $n$ , and where unimodular means that the entries are complex numbers that lie on the unit circle. In other words, any two distinct rows (or columns) of  $H$  are complex orthogonal. Complex Hadamard matrices have various applications in mathematics ranging from coding [5] and operator theory [12] to harmonic analysis [8]. They also play a crucial rôle in quantum information theory, for construction of quantum teleportation and dense coding schemes [19], and they are strongly related to mutually unbiased bases (MUBs) [21]. In this paper we are primarily concerned with  $n \times n$  Butson-type Hadamard matrices [2], denoted by  $BH(q, n)$ , which are complex Hadamard matrices composed of  $q$ th roots of unity. The notations  $BH(2, n)$  and  $BH(4, n)$  correspond to the real Hadamard matrices and quaternary complex Hadamard matrices, respectively.

Recently there has been a renewed interest in  $BH(q, n)$  matrices. In particular, they have been used as starting-point matrices for constructing parametric families of complex Hadamard matrices for various  $q$  and  $n$ . In a series of papers Diţă [4], Szöllősi [15], and Tadej and Życzkowski [17] obtained new, previously unknown parametric families of complex Hadamard matrices from  $BH(q, n)$  matrices. The parametrization of a  $BH(q, n)$  matrix is an operation where some of the matrix entries are replaced by infinitely differentiable, or smooth, functions mapping a real-valued or complex-valued argument vector into the set of unimodular complex numbers. Any assignment of values to the parameters yields a complex Hadamard matrix and some assignments produce  $BH(q, n)$  matrices. The parametrization of

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*Date:* February 2012, preprint.

P.H.J. L. was supported by the Academy of Finland, Grants No. 132122.

F. Sz. was supported by the Hungarian National Research Fund OTKA K-77748.

P.R.J. Ö. was supported by the Academy of Finland, Grants No. 130142 and No. 132122.

the  $BH(q, n)$  matrices makes it possible to escape equivalence classes and therefore to collect many inequivalent matrices into a single parametric family. Also the switching operation [16], a well-known technique in design theory, has this property.

The existence of  $BH(q, n)$  matrices is wide open in general. Even the simplest case for  $q = 2$  is undecided, as the famous Hadamard conjecture, stating that  $BH(2, 4k)$  matrices exist for every positive integer  $k$ , remains elusive despite continuous efforts [3]. Real Hadamard matrices, or  $BH(2, n)$  matrices, are completely classified up to  $n = 28$ , and there has been some promising advance in enumerating the case  $n = 32$  very recently (see [7] and the references therein). For other values of  $q$  we have some constructions [2] and some non-existence results [20]. Harada, Lam and Tonchev classified all  $16 \times 16$  generalized Hadamard matrices over groups of order 4 and obtained new examples of  $BH(4, 16)$  matrices [5]. This particular result motivated the authors to investigate the existence of  $BH(4, n)$  matrices in more general and to start enumerating and classifying them for small  $n$ . The census of quaternary complex Hadamard matrices up to order 8 were carried out in [14]. The aim of this paper is to continue that work, and give a complete classification of the  $BH(4, n)$  matrices up to orders  $n = 14$ .

This work has two major parts: first, we classify all  $BH(q, n)$  matrices using computer-aided methods, and secondly, we define parametric families of complex Hadamard matrices by introducing parameters to the  $BH(q, n)$  matrices. Parametric families offer a compact way of representing a large number of  $BH(q, n)$  matrices, and they also yield information about complex Hadamard matrices.

In the previous work [14] the authors collected all  $BH(4, n)$  matrices contained in various parametric families from the existing literature and then confirmed with an exhaustive computer search that those are the only examples. Here the situation is exactly the opposite, as it turns out that almost all  $BH(4, n)$  matrices of orders  $n = 12$  and  $14$  found in the current work by the computer search are previously unknown. Therefore we are facing the inverse problem: we need to encode a given collection of  $BH(4, n)$  matrices by parametric families.

The outline of the paper is as follows. In Section 2 we give some basic definitions and results that are used in later sections. Then in Section 3 we recall the computer-aided classification method of difference matrices from [10] and highlight the main differences and necessary modifications required for our purposes. In Section 4 we introduce a new method for parametrizing complex Hadamard matrices and recall some relevant materials from the existing literature. Finally, in Sections 5, 6, and 7 we present the classification of  $BH(4, n)$  matrices for  $n = 10, 12$ , and  $14$ , respectively. To improve the readability of our paper a rather long list of complex Hadamard matrices has been moved from the main text to Appendix A. The matrices the authors obtained are also available in an electronic format in the on-line repository [9] as a supplement. Throughout this paper we adopt the notations from [17] for known complex Hadamard matrices, such as  $D_{10}^{(3)}(a, b, c)$ , etc.

## 2. PRELIMINARIES

Two  $BH(q, n)$  matrices are equivalent if the first matrix can be transformed into the second one by permuting the order of its rows and columns and multiplying a row or a column by some  $q$ th root of unity. The automorphism group of a  $BH(q, n)$  matrix  $H$  is the group of pairs of monomial matrices  $(P, Q)$  such that  $H = PHQ$ ; a monomial matrix is an  $n \times n$  matrix having a single nonzero entry in each row and column, these nonzero entries being complex  $q$ th roots of unity. Note that the automorphism group depends on the choice of  $q$ .

We employ invariants for finding the equivalence class of a  $\text{BH}(q, n)$  matrix in a computationally efficient way. The determinant of a  $k \times k$  submatrix of matrix  $H$  is called a vanishing minor, or a zero minor, of  $H$  if the determinant is zero. Throughout this paper we shall repeatedly use the following

**Lemma 2.1.** *Let  $H$  be a complex Hadamard matrix of order  $n$  and  $2 \leq k \leq n - 2$  be an integral number. Then the number of  $k \times k$  vanishing minors of  $H$  is invariant, up to equivalence.*

The number of  $k \times k$  vanishing minors is just a special case of a more powerful invariant, the fingerprint [15], but it is sufficient for our purposes.

The  $q$ -rank, or  $\mathbb{Z}_q$ -rank, of an integral matrix  $L$  of order  $n$  is the smallest positive integer  $r$  such that there are integral matrices  $S$  and  $T$  of orders  $n \times r$  and  $r \times n$  respectively, such that  $ST \equiv L \pmod{q}$ . The  $\mathbb{Z}_q$ -rank of a  $\text{BH}(q, n)$  matrix  $H$  is the  $\mathbb{Z}_q$ -rank of the  $(0, 1, \dots, q-1)$ -matrix  $L$  for which  $H = \text{EXP}\left(\frac{2\pi i}{q} L\right)$ , where  $\text{EXP}$  denotes the entry-wise exponentiation function. It is easy to see that the  $\mathbb{Z}_q$ -rank is, again, invariant up to equivalence. A class of  $\text{BH}(q, n)$  matrices with small  $\mathbb{Z}_q$ -rank has some interesting applications in harmonic analysis [8].

As it seems that a complex Hadamard matrix  $H$  shares every important property with its hermitian  $H^*$ , complex conjugate  $\overline{H}$  and transpose  $H^T$ , we introduce the concept of ACT-equivalence. Two complex Hadamard matrices  $H$  and  $K$  are called ACT-equivalent, if  $H$  is equivalent to at least one of  $K$ ,  $K^*$ ,  $\overline{K}$  or  $K^T$ . The concept of this refined equivalence simplifies the presentation of our results as we can avoid unnecessary repetitions in our summarizing tables [9].

### 3. CONSTRUCTING BUTSON-TYPE HADAMARD MATRICES

The computer-aided methods employed in the classification of Butson-type Hadamard matrices in this work are very similar to the methods used for the classification of difference matrices over cyclic groups [10]. Therefore, we give here only a summary of the relevant ideas and describe in detail only the points where this work differs from the work done on difference matrices.

We perform an exhaustive computer search of the space of all Butson-type Hadamard matrices  $\text{BH}(q, n)$  with the given parameters  $q = 4$  and  $n = 10, 12, 14$ . Because the size of the search space grows very quickly as the parameters are increased, we have to prune most of the search space and we do this by considering only inequivalent matrices.

An  $m \times n$  matrix over  $q$ th roots of unity is said to be a candidate  $\text{BH}(q, n)$  matrix, denoted by  $\text{BH}(q, m, n)$ , if the rows are orthogonal as in a  $\text{BH}(q, n)$  matrix, but the number of rows  $m \leq n$ . A  $\text{BH}(q, m, n)$  matrix is dephased (or normalized) if the first row and the first column contain only the value 1. Since every  $\text{BH}(q, m, n)$  matrix is equivalent to a dephased one, only these matrices are considered in the computer search.

The dephased candidate  $\text{BH}(q, m, n)$  matrices are organized into a tree where the child nodes of a  $\text{BH}(q, m, n)$  are all  $\text{BH}(q, m+1, n)$  that are obtained by appending a row to the node, cf. [10, Definition 2.6]. An exhaustive search of this tree is performed by weak canonical augmentation [6, Section 4.2.3], which has the advantage over the simpler breadth-first search used in [10] that it can be performed easily in parallel.

Consider a tree of dephased candidate  $\text{BH}(q, m, n)$  matrices. Let  $p(X)$  denote the parent of the node  $X$  in the tree. Every node  $X$  in the tree has a finite sequence of ancestors from

which it has been constructed by appending a row to it:

$$(1) \quad X, p(X), p(p(X)), p(p(p(X))), \dots$$

In the tree each  $X$  occurs only once but there are typically many nodes  $Y$  with  $X \cong Y$ . Such a node  $Y$  has also a sequence of ancestors from which it has been constructed:

$$(2) \quad Y, p(Y), p(p(Y)), p(p(p(Y))), \dots$$

Even though  $X \cong Y$  the ancestor sequences (1) and (2) need not consist of the same nodes up to equivalence. This means that the sequences (1) and (2) can be distinct on the level of equivalence classes even if  $X \cong Y$ . The main idea is now to exploit such differences among equivalent nodes in rejecting equivalent nodes.

Let  $T_{nr}$  denote the set of all non-root nodes in the search tree  $T$ . Associate with every object  $X \in T_{nr}$  a weak canonical parent  $w(X) \in T$  such that the following property holds:

$$(3) \quad \text{for all } X, Y \in T_{nr} \text{ it holds that } X \cong Y \text{ implies } w(X) \cong w(Y).$$

The function  $w$  defines for every non-root node  $X$  a sequence of objects analogous to (1):

$$(4) \quad X, w(X), w(w(X)), w(w(w(X))), \dots$$

Because of (3), any two equivalent objects have identical sequences (4) on the level of equivalence classes of matrices. When the search tree is traversed in depth-first order, a node  $X$  and the subtree rooted at it is considered only if it has been constructed in the canonical way specified by (4); that is, every matrix in the ancestor sequence (1) should be equivalent to the matrix in the corresponding position in the sequence (4). By (3) we obtain

$$p(X) \cong w(X) \cong w(Y) \cong p(Y)$$

In other words, equivalent matrices generated by weak canonical augmentation have equivalent parent matrices, and assuming that the same holds for the parents, this implies that equivalent matrices must be siblings in the search tree. This reduces the size of the search tree dramatically.

The problem of checking equivalence of  $BH(q, m, n)$  matrices is solved by transforming it into a corresponding graph isomorphism problem in exactly the same way as was done with difference matrices over cyclic groups in [10, Section 3]. Each  $BH(q, m, n)$  matrix is mapped to a directed graph, called the equivalence graph of the matrix. Two  $BH(q, m, n)$  matrices are equivalent if and only if their equivalence graphs are isomorphic graphs. The definition of an equivalence graph is obtained from the definition of a difference matrix graph [10, Definition 3.2] by considering  $q$ th roots of unity as a cyclic group where the operation is complex multiplication.

The software package NAUTY [11] is used for checking the isomorphism of equivalence graphs. For each equivalence graph NAUTY calculates a graph, called the canonical graph, which has the property that two equivalence graphs are isomorphic if and only if they have the same canonical graph. This implies that two  $BH(q, m, n)$  matrices are equivalent if and only if the canonical graphs of the equivalence graphs of the matrices are the same. We define the canonical graph of a matrix  $X$  as the canonical graph of the equivalence graph of  $X$ .

Transforming a matrix to a graph yields also a weak canonical function  $w$  having the property (3). Let  $P(X)$  be the set of matrices obtained by removing a row from a matrix  $X \in T_{nr}$  and let  $\leq_g$  denote a total order on the set of canonical graphs. We define  $w(X)$  as

the matrix  $Y \in P(X)$  which has the smallest canonical graph under  $\leq_g$  among the canonical graphs of matrices in  $P(X)$ .

The definition of equivalence of  $\text{BH}(q, n)$  matrices given in Section 2 agrees with the equivalence relation  $\cong^*$  defined in [10, Section 2] and the equivalence inducing group  $E^*$  in [10, Section 5].

To get confidence in the computational results, we perform a consistency check by counting the number of candidate and complete  $\text{BH}(q, n)$  matrices in two different ways. In the first method the sizes of the equivalence classes of  $\text{BH}(q, m, n)$  matrices are obtained from inequivalent  $\text{BH}(q, m, n)$  matrices and the automorphism groups of their canonical graphs via the orbit-stabilizer theorem. The second method determines the number of  $\text{BH}(q, m, n)$  matrices from the number of  $\text{BH}(q, m - 1, n)$  matrices and number of ways these smaller matrices can be augmented to yield a  $\text{BH}(q, m, n)$  matrix. This double-counting method is the same as in [10, Section 5] except that the analytical formula for  $2 \times c$  difference matrices and the reduction used for  $3 \times c$  difference matrices are not valid for  $\text{BH}(q, 2, n)$  and  $\text{BH}(q, 3, n)$  matrices, respectively. With  $\text{BH}(q, n)$  matrices the double-counting starts one level earlier with  $\text{BH}(q, 1, n)$  matrices, which all belong to the same equivalent class of size  $q^n$ .

The exhaustive computer search yields all inequivalent  $\text{BH}(q, n)$  matrices and the automorphism group for each matrix.

#### 4. PARAMETRIZING COMPLEX HADAMARD MATRICES

Once a complete set of inequivalent  $\text{BH}(4, n)$  matrices were obtained the authors investigated whether these matrices lead to parametric families of complex Hadamard matrices. We recall the following basic method to introduce free parameters into complex Hadamard matrices of even order [15, Lemma 3.4].

**Lemma 4.1.** *Let  $H$  be a dephased complex Hadamard matrix of order  $n$  and suppose that there exist a pair of rows in  $H$ , say  $u$  and  $v$ , such that for every  $i = 1, 2, \dots, n$ ,  $u_i^2 = v_i^2$ . Then for all such  $i$  for which  $u_i + v_i = 0$  replace  $u_i$  with  $\alpha u_i$  and  $v_i$  with  $\alpha v_i$ , where  $\alpha$  is a unimodular complex number to obtain a one-parameter family of complex Hadamard matrices  $H(\alpha)$ .*

Lemma 4.1 is a general method for introducing affine parameters into real Hadamard matrices, irrespectively of their order, and it can be applied to various other matrices as well. Unfortunately, however, it cannot be applied to  $\text{BH}(4, n)$  matrices when  $n \equiv 2 \pmod{4}$ . In the following we present a method working for some of these  $\text{BH}(4, n)$  matrices.

Let us denote by  $1^m$  the all-1 row vector of length  $m$ , and let  $\langle \cdot, \cdot \rangle$  denote the standard inner product in  $\mathbb{C}^d$  with the convention that it is linear in the first, and conjugate linear in its second argument.

**Theorem 4.2.** *Let  $H$  be a dephased complex Hadamard matrix with the following block structure*

$$H = \begin{bmatrix} 1 & 1 & 1 & 1^p & 1^q \\ 1 & a & b & x & y \\ 1 & b & a & x & -y \\ (1^r)^T & z^T & z^T & A & B \\ (1^s)^T & w^T & -w^T & C & D \end{bmatrix},$$

where  $a$  and  $b$  are arbitrary unimodular numbers. Then, after replacing the row vectors  $y$  with  $\alpha y$  and  $w$  with  $\bar{\alpha}w$  we obtain a one-parameter family of complex Hadamard matrices  $H(\alpha)$  where  $\alpha$  is unimodular. If, in addition,  $b = a$  we can continue by replacing  $w$  in  $H(\alpha)$  with  $\alpha\beta w$  to obtain a two-parameter family of complex Hadamard matrices  $H(\alpha, \beta)$  where  $\alpha$  and  $\beta$  are unimodular.

*Proof.* We need to show that the rows of  $H(\alpha, \beta)$  are pairwise orthogonal. From the orthogonality of the first three rows of  $H$  we get

$$\left. \begin{aligned} \langle [1, 1, 1, 1^p, 1^q], [1, a, b, x, y] \rangle &= 0 \\ \langle [1, 1, 1, 1^p, 1^q], [1, b, a, x, -y] \rangle &= 0 \end{aligned} \right\} \implies \langle 1^q, y \rangle = 0,$$

and

$$\begin{aligned} 0 = \langle [1, a, b, x, y], [1, b, a, x, -y] \rangle &= 1 + a\bar{b} + b\bar{a} + p - q \\ &= \langle [1, a, b, x, \alpha y], [1, a, b, x, -\alpha y] \rangle, \end{aligned}$$

and hence the first three rows of  $H(\alpha, \beta)$  are pairwise orthogonal. Similarly, it is easily seen that the rest of the rows (beyond the first three) are pairwise orthogonal within themselves. Additionally, the first row is trivially orthogonal to all further rows. Therefore it remains to be seen that the second and third rows of  $H(\alpha, \beta)$  are orthogonal to all rows below them.

We show first that they are orthogonal to the rows which are of type  $[1, z_i, z_i, A_i, B_i]$ ,  $i = 1, \dots, r$ . In the original matrix  $H$  (i.e., prior to parametrizing) we have

$$(5) \quad 1 + z_i(\bar{a} + \bar{b}) + \langle A_i, x \rangle + \langle B_i, y \rangle = 0,$$

$$(6) \quad 1 + z_i(\bar{a} + \bar{b}) + \langle A_i, x \rangle - \langle B_i, y \rangle = 0,$$

and hence  $\langle B_i, y \rangle = 0$  for every  $i = 1, \dots, r$ . It follows that after parametrization equations (5) and (6) remain valid.

We proceed by proving that rows that are of type  $[1, w_i, -w_i, C_i, D_i]$ ,  $i = 1, \dots, s$ , after parametrization, are orthogonal to the second and third row of  $H(\alpha, \beta)$ . Again, in the original matrix  $H$  we have

$$(7) \quad 1 + w_i\bar{a} - w_i\bar{b} + \langle C_i, x \rangle + \langle D_i, y \rangle = 0,$$

$$(8) \quad 1 - w_i\bar{a} + w_i\bar{b} + \langle C_i, x \rangle - \langle D_i, y \rangle = 0,$$

and hence  $\langle C_i, x \rangle = -1$  for every  $i = 1, \dots, s$ . It follows, that (7) and (8) are valid, provided that

$$(9) \quad w_i\bar{a} - w_i\bar{b} + \langle D_i, y \rangle = 0$$

holds and therefore (7) and (8) remains true, after parametrization. If, in addition,  $b = a$ , then  $\langle D_i, y \rangle = 0$  for every  $i = 1, \dots, s$ , and hence (9) holds, independently of the scalar factor in  $w$ .

The one-parameter family  $H(\alpha)$  can be considered as  $H(\alpha, \bar{\alpha})$ . The equations (5) - (8) hold as the condition  $a = b$  is not required for them. From the original matrix  $H$  we get

$$w_i\bar{a} - w_i\bar{b} + \langle D_i, y \rangle = 0 \implies \bar{\alpha}w_i\bar{a} - \bar{\alpha}w_i\bar{b} + \langle D_i, \alpha y \rangle = 0,$$

and (9) holds for  $H(\alpha, \bar{\alpha})$  also when  $a \neq b$ . □

*Remark 4.3.* If  $a$  and  $b$  are as in Theorem 4.2, then it is easy to see that the real part of  $a\bar{b}$  is an integral number, and therefore  $b \in \pm a \cdot \{1, \omega, \omega^2, \mathbf{i}\}$ , where  $\omega$  is a primitive complex third root of unity. Therefore one hopes to apply the parametrizing scheme described for complex Hadamard matrices with fourth and/or sixth roots of unity.

Theorem 4.2 describes a local property of the complex Hadamard matrix  $H$ . Its conditions can be fairly easily checked, even by hand, and it can be implemented as a computer program to construct infinite families automatically.

It is natural to ask how many degrees of freedom (i.e., independent, smooth parameters) can be introduced into a given complex Hadamard matrix. We recall the following fundamental concept from [17]: The defect  $d(H)$  of an  $n \times n$  complex Hadamard matrix  $H$  reads  $d(H) = m - 2n + 1$ , where  $m$  is the dimension of the solution space of the following real linear system with respect to the matrix variable  $R \in \mathbb{R}^{n \times n}$ :

$$(10) \quad \sum_{k=1}^n H_{i,k} \overline{H}_{j,k} (R_{i,k} - R_{j,k}) = 0, \quad 1 \leq i < j \leq n.$$

In what follows we motivate the formula (10). Consider a dephased complex Hadamard matrix  $H$  and a phasing matrix  $R$  whose first row and column is 0, and all other entries are a real variables. Now consider the matrix  $K := H \circ \text{EXP}(\mathbf{i}R)$ , where  $\circ$  is the entry-wise product and  $\text{EXP}$  is the entry-wise exponential function, respectively. The resulting matrix  $K$  is the most general parametrized matrix, stemming from  $H$ . Note that  $K$  is unimodular, but not necessarily complex Hadamard. In order to ensure this latter condition we spell out the orthogonality conditions of  $K$ , obtaining

$$(11) \quad \langle K_i, K_j \rangle = \sum_{k=1}^n H_{i,k} \overline{H}_{j,k} \mathbf{e}^{\mathbf{i}(R_{i,k} - R_{j,k})} = 0, \quad 1 \leq i < j \leq n,$$

which, after linearizing the exponential function (i.e., replacing it with its first order Taylor expansion) leads to (10). Therefore those phasing matrices  $R$  satisfying the real linear system (10) lead to parametric families of complex Hadamard matrices in a neighborhood of the initial matrix  $H$ , up to first order; however, (11) is far more restrictive further decreasing the number of parameters in  $R$  in general.

Note that the degree of freedom  $m$  in the defect of an  $n \times n$  matrix is decreased by  $2n - 1$  as this many parameters can always be introduced into a complex Hadamard matrix via multiplication by unitary diagonal matrices. This operation, however, does not yield new complex Hadamard matrices, up to equivalence, and therefore only dephased families are considered. Matrices that cannot be parametrized in any other way are called isolated. The most important properties of the defect are summarized in the following result from [17, 18].

**Proposition 4.4.** *Let  $H$  be a complex Hadamard matrix. Then*

- (a)  $d(H)$  is an invariant, up to equivalence, moreover  $d(H) = d(H^*) = d(\overline{H}) = d(H^T)$ ;
- (b) the number of smooth parameters which can be introduced into  $H$  is at most  $d(H)$ ;
- (c) if  $d(H) = 0$  then  $H$  is isolated amongst all  $n \times n$  complex Hadamard matrices.

Note that part (c) does not require the smoothness condition and as a result it does not follow from part (b).

*Remark 4.5.* It might be possible to use (10) to extend known affine families with further parameters as follows. Suppose that  $H(\alpha)$  is a dephased parametric family of complex

Hadamard matrices. Let  $k > 1$  be an integer and evaluate  $H(\alpha)$  at  $k$  random points to obtain a series of dephased complex Hadamard matrices  $H_1, H_2, \dots, H_k$ . Now consider a phasing matrix  $R$  (whose entries are real variables) and the resulting general parametric matrices  $K_i = H_i \circ \text{EXP}(\text{iR})$  and the corresponding  $k$  system of linear equations given by (10) for  $i = 1, \dots, k$ . A common solution of this  $k$  system, which should be further refined by (11), might result in families of complex Hadamard matrices with more free parameters than the initial one  $H(\alpha)$ .

It is easy to see that a dephased complex Hadamard matrix is equivalent to a  $\text{BH}(q, n)$  matrix if and only if all of its entries are some  $q$ th root of unity; consequently it is easy to obtain all  $\text{BH}(q, n)$  matrices which are members of some parametric family  $H(\alpha)$ , stemming from the starting point  $\text{BH}(q, n)$  matrix  $H$ .

## 5. THE QUATERNARY COMPLEX HADAMARD MATRICES OF ORDER 10

In this section we report on our findings regarding  $\text{BH}(4, 10)$  matrices.

**Theorem 5.1.** *There are exactly 10  $\text{BH}(4, 10)$  matrices, up to equivalence, forming 7 ACT-equivalence classes.*

Now we present affine families containing all inequivalent  $\text{BH}(4, n)$  matrices. The first family,

$$D_{10}^{(3)}(a, b, c) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -\text{i}a\bar{b} & -\text{i}a & -\text{i}\bar{c} & -\text{i} & \text{i}\bar{c} & \text{i}a & \text{i}a\bar{b} & \text{i} \\ 1 & -\text{i}b\bar{a} & -1 & \text{i}b & \text{i}\bar{c} & -\text{i} & -\text{i}\bar{c} & -\text{i}b & \text{i} & \text{i}b\bar{a} \\ 1 & -\text{i}\bar{a} & \text{i}\bar{b} & -1 & -\text{i} & \text{i} & -\text{i} & \text{i} & -\text{i}\bar{b} & \text{i}\bar{a} \\ 1 & -\text{i}c & \text{i}c & -\text{i} & -1 & \text{i} & \text{i} & -\text{i} & \text{i}c & -\text{i}c \\ 1 & -\text{i} & -\text{i} & \text{i} & \text{i} & -1 & \text{i} & \text{i} & -\text{i} & -\text{i} \\ 1 & \text{i}c & -\text{i}c & -\text{i} & \text{i} & \text{i} & -1 & -\text{i} & -\text{i}c & \text{i}c \\ 1 & \text{i}\bar{a} & -\text{i}\bar{b} & \text{i} & -\text{i} & \text{i} & -\text{i} & -1 & \text{i}\bar{b} & -\text{i}\bar{a} \\ 1 & \text{i}b\bar{a} & \text{i} & -\text{i}b & \text{i}\bar{c} & -\text{i} & -\text{i}\bar{c} & \text{i}b & -1 & -\text{i}b\bar{a} \\ 1 & \text{i} & \text{i}a\bar{b} & \text{i}a & -\text{i}\bar{c} & -\text{i} & \text{i}\bar{c} & -\text{i}a & -\text{i}a\bar{b} & -1 \end{bmatrix},$$

has been obtained in [15]. It contains three out of the seven ACT-classes, namely  $D_{10}^{(3)}(1, 1, 1)$ ,  $D_{10}^{(3)}(1, 1, \text{i})$  and  $D_{10}^{(3)}(1, \text{i}, \text{i})$ . Note that its orbit can be constructed by repeatedly using Theorem 4.2.

The second family reads

$$N_{10B}^{(3)}(a, b, c) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -\text{i} & -1 & -1 & \text{i} \\ 1 & a & c & -\text{i}c & -a & -1 & -c & -\text{i}abc & \text{i}abc & \text{i}c \\ 1 & -a & -\text{i}c & c & a & -1 & -c & \text{i}abc & -\text{i}abc & \text{i}c \\ 1 & -\text{i} & -\text{i}\bar{a}c & \text{i}\bar{a}c & -1 & 1 & \text{i} & bc & -bc & -1 \\ 1 & \text{i}\bar{b} & -\text{i}\bar{a}bc & \text{i}\bar{a}bc & -\text{i}\bar{b} & -1 & c & -c & \text{i}c & -\text{i}c \\ 1 & -\text{i}\bar{b} & \text{i}\bar{a}bc & -\text{i}\bar{a}bc & \text{i}\bar{b} & -1 & c & \text{i}c & -c & -\text{i}c \\ 1 & -1 & \text{i}\bar{a}c & -\text{i}\bar{a}c & -\text{i} & 1 & \text{i} & -bc & bc & -1 \\ 1 & \text{i} & -c & -c & \text{i} & -\text{i} & -1 & c & c & -\text{i} \\ 1 & -1 & \text{i}c & \text{i}c & -1 & \text{i} & -\text{i} & -\text{i}c & -\text{i}c & 1 \end{bmatrix},$$



whose one-parametric subfamily  $N_{10B}^{(1)}(a) = N_{10B}^{(3)}(a, 1, 1)$  was reported in [1]. This matrix contains two ACT-classes, namely  $N_{10B}^{(3)}(1, 1, 1)$  and  $N_{10B}^{(3)}(\mathbf{i}, 1, 1)$ , so despite the two-degree extension, which was obtained by a repeated application of Theorem 4.2, no previously unknown BH(4, 10) matrix surfaced.

The two remaining matrices can be obtained from complex Golay sequences [3], and they belong to the family

$$G_{10}^{(1)}(a) = \left[ \begin{array}{ccccc|ccccc} 1 & a & a^2 & \mathbf{i}a^3 & -\mathbf{i}a^4 & 1 & \mathbf{i}a & -\mathbf{i}a^2 & -a^3 & \mathbf{i}a^4 \\ -\mathbf{i}a^4 & 1 & a & a^2 & \mathbf{i}a^3 & \mathbf{i}a^4 & 1 & \mathbf{i}a & -\mathbf{i}a^2 & -a^3 \\ \mathbf{i}a^3 & -\mathbf{i}a^4 & 1 & a & a^2 & -a^3 & \mathbf{i}a^4 & 1 & \mathbf{i}a & -\mathbf{i}a^2 \\ a^2 & \mathbf{i}a^3 & -\mathbf{i}a^4 & 1 & a & -\mathbf{i}a^2 & -a^3 & \mathbf{i}a^4 & 1 & \mathbf{i}a \\ a & a^2 & \mathbf{i}a^3 & -\mathbf{i}a^4 & 1 & \mathbf{i}a & -\mathbf{i}a^2 & -a^3 & \mathbf{i}a^4 & 1 \\ \hline 1 & -\mathbf{i}\bar{a}^4 & -\bar{a}^3 & \mathbf{i}\bar{a}^2 & -\mathbf{i}\bar{a} & -1 & -\mathbf{i}\bar{a}^4 & \mathbf{i}\bar{a}^3 & -\bar{a}^2 & -\bar{a} \\ -\mathbf{i}\bar{a} & 1 & -\mathbf{i}\bar{a}^4 & -\bar{a}^3 & \mathbf{i}\bar{a}^2 & -\bar{a} & -1 & -\mathbf{i}\bar{a}^4 & \mathbf{i}\bar{a}^3 & -\bar{a}^2 \\ \mathbf{i}\bar{a}^2 & -\mathbf{i}\bar{a} & 1 & -\mathbf{i}\bar{a}^4 & -\bar{a}^3 & -\bar{a}^2 & -\bar{a} & -1 & -\mathbf{i}\bar{a}^4 & \mathbf{i}\bar{a}^3 \\ -\bar{a}^3 & \mathbf{i}\bar{a}^2 & -\mathbf{i}\bar{a} & 1 & -\mathbf{i}\bar{a}^4 & \mathbf{i}\bar{a}^3 & -\bar{a}^2 & -\bar{a} & -1 & -\mathbf{i}\bar{a}^4 \\ -\mathbf{i}\bar{a}^4 & -\bar{a}^3 & \mathbf{i}\bar{a}^2 & -\mathbf{i}\bar{a} & 1 & -\mathbf{i}\bar{a}^4 & \mathbf{i}\bar{a}^3 & -\bar{a}^2 & -\bar{a} & -1 \end{array} \right].$$

The matrices  $G_{10}^{(1)}(1)$  and  $G_{10}^{(1)}(-1)$  are inequivalent from all previously considered examples. We note here that the family  $G_{10}^{(1)}$  implies the existence of an infinite family of triplets of pairwise mutually unbiased bases in  $\mathbb{C}^{10}$  by a construction of Zauner [21].

**Theorem 5.2.** *Each member of the ACT-equivalence classes of BH(4, 10) matrices can be obtained from three partially overlapping infinite affine parametric families of complex Hadamard matrices. These matrices, up to ACT-equivalence, can be recognized by the number of  $3 \times 3$  vanishing minors they contain.*

It turns out that all BH(4, 10) matrices appear in the existing literature, however, the parametric families  $N_{10B}^{(3)}(a, b, c)$  and  $G_{10}^{(1)}(a)$  are considered here for the first time. The authors believe that these families contain new, previously unknown BH( $q$ , 10) matrices for some  $q > 4$ . Table 1 summarizes the properties of BH(4, 10) matrices for which the legend is as follows: the column “ACT” describes if the matrix is equivalent to its adjoint, conjugate or transpose, respectively, while the column “HBS” indicates if it is equivalent to a Hermitian matrix, if contains a  $n/2 \times n/2$  sub-Hadamard matrix (i.e., it comes from a doubling construction, see [12]) or if is equivalent to a symmetric matrix, respectively; the rest of the column headings speak for themselves.

## 6. THE QUATERNARY COMPLEX HADAMARD MATRICES OF ORDER 12

The main result concerning BH(4, 12) matrices is the following.

**Theorem 6.1.** *There are exactly 319 BH(4, 12) matrices, up to equivalence, forming 167 ACT-equivalence classes.*

Because all but four representatives of the ACT-equivalence classes contain a pair of real rows, almost all matrices belong to some parametric families through Lemma 4.1. It turned out that the exceptional four matrices can be parametrized as well by solving (10) and (11), and they can be described by two parametric families (consult the families  $L_{12A}^{(1)}(a)$  and  $L_{12B}^{(1)}(a)$  in Appendix A). Therefore every BH(4, 12) matrix belongs to some parametric

TABLE 1. Summary of the BH(4, 10) matrices

ACT class	Equiv. classes	Family, coordinates	ACT	HBS	Auto. order	Defect	Orbit	$\mathbb{Z}_4$ rank	Invariant ( $3 \times 3$ )
1	1	$D_{10}^{(3)}(1, 1, \mathbf{i})$	YYY	Y–Y	64	10	3	9	2032
2	2, 5	$N_{10B}^{(3)}(\mathbf{i}, 1, 1)$	NNY	N–Y	192	11	3	9	2496
3	3	$D_{10}^{(3)}(1, 1, 1)$	YYY	Y–Y	2880	16	3	9	3600
4	4	$D_{10}^{(3)}(1, \mathbf{i}, \mathbf{i})$	YYY	Y–Y	32	12	3	9	2080
5	6, 8	$N_{10B}^{(3)}(1, 1, 1)$	NNY	N–Y	64	7	3	9	1568
6	7	$G_{10}^{(1)}(1)$	YYY	Y–Y	20	8	1	9	1580
7	9, 10	$G_{10}^{(1)}(-1)$	NYN	N–N	80	8	1	9	1600

family. We, however, tried to minimize the number of parametric families required for the presentation of all BH(4, 12) matrices by attempting to describe families capturing essentially different properties of these matrices. We managed to describe the matrices with the aid of 23 families. In what follows we display four of the 23 families containing altogether a considerable number of distinct ACT-classes.

The first family, stemming from the real Hadamard matrix  $H_{12}$ , has 10 free parameters, and is the largest known affine family of order 12. It contains 36 distinct ACT-classes.

$$H_{12B}^{(10)}(a, b, c, d, e, f, g, h, i, j) =$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & a & -a & c & -c & e & e & -e & -e \\ 1 & 1 & -1 & -1 & -a & a & c & -c & -e & -e & e & e \\ 1 & 1 & b & -b & -1 & -1 & -c & c & g & -g & h & -h \\ 1 & 1 & -b & b & -1 & -1 & -c & c & -g & g & -h & h \\ 1 & -1 & d & d & -d & -d & cd & -cd & fg\bar{b} & -fg\bar{b} & -fh\bar{b} & fh\bar{b} \\ 1 & -1 & -d & -d & d & d & -cd & cd & fg\bar{b} & -fg\bar{b} & -fh\bar{b} & fh\bar{b} \\ 1 & -1 & f & -f & i & -i & -ei\bar{a} & -ei\bar{a} & -fg\bar{b} & fg\bar{b} & ei\bar{a} & ei\bar{a} \\ 1 & -1 & f & -f & -i & i & ei\bar{a} & ei\bar{a} & -fg\bar{b} & fg\bar{b} & -ei\bar{a} & -ei\bar{a} \\ 1 & -1 & -f & f & j & -j & ej\bar{a} & ej\bar{a} & -ej\bar{a} & -ej\bar{a} & fh\bar{b} & -fh\bar{b} \\ 1 & -1 & -f & f & -j & j & -ej\bar{a} & -ej\bar{a} & ej\bar{a} & ej\bar{a} & fh\bar{b} & -fh\bar{b} \end{bmatrix}$$

The second matrix has 8 parameters and contains 43 distinct ACT-equivalence classes of the BH(4, 12) matrices.

$$H_{12C}^{(8)}(a, b, c, d, e, f, g, h) =$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & c & 1 & 1 & a & -a & -c & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -b & b & -a & a & -1 & e & d & -d & -e \\ 1 & 1 & -f & -1 & -1 & f & f & -f & -ef & -d & d & ef \\ 1 & 1 & f & -1 & -1 & -f & -f & f & ef & -d & d & -ef \\ 1 & 1 & -c & b & -b & -1 & -1 & c & -e & d & -d & e \\ 1 & -1 & cgh & bgh & -bgh & -agh & agh & -cgh & h & -h & -h & h \\ 1 & -1 & -cgh & -bgh & bgh & agh & -agh & cgh & h & -h & -h & h \\ 1 & -1 & ch & -bh & bh & -h & -h & -ch & -eh & h & h & eh \\ 1 & -1 & h & -h & -h & h & h & h & -h & dh & -dh & -h \\ 1 & -1 & -ch & h & h & -ah & ah & ch & -h & -dh & dh & -h \\ 1 & -1 & -h & bh & -bh & ah & -ah & -h & eh & h & h & -eh \end{bmatrix}.$$

The families  $H_{12B}^{(10)}$  and  $H_{12C}^{(8)}$  intersect in 19 ACT-classes and they represent 60 distinct ACT-classes. The next family comes from Diță's general method [4] and all members of it contain a  $6 \times 6$  sub-Hadamard matrix  $D_6^{(1)}(a)$ , see [16]:

$$D_{12}^{(7)}(a, b, c, d, e, f, g) = \begin{bmatrix} D_6^{(1)}(a) & \text{Diag}(1, b, c, d, e, f)D_6^{(1)}(g) \\ D_6^{(1)}(a) & -\text{Diag}(1, b, c, d, e, f)D_6^{(1)}(g) \end{bmatrix}.$$

The family  $D_{12}^{(7)}$  contains 20 ACT-equivalence classes, none of which is equivalent to the previously considered 60 classes.

The fourth family, containing 22 ACT-classes, of which 21 is distinct from all previously discussed reads

$$X_{12}^{(7)}(a, b, c, d, e, f, g) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \mathbf{i} & \mathbf{i} & -1 & -1 & -1 & -1 & -\mathbf{i} & -\mathbf{i} \\ 1 & 1 & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{ia} & \mathbf{ib} & -\mathbf{ib} & -\mathbf{ia} & -1 & -1 \\ 1 & 1 & -1 & -1 & e & -e & a & -\mathbf{ib} & \mathbf{ib} & -a & f & -f \\ 1 & 1 & -1 & -1 & -e & e & -\mathbf{ia} & b & -b & \mathbf{ia} & -f & f \\ 1 & 1 & -\mathbf{i} & -\mathbf{i} & -1 & -1 & -a & -b & b & a & \mathbf{i} & \mathbf{i} \\ 1 & -1 & cg & -cg & g & -g & -ag\bar{e} & bg\bar{e} & -bg\bar{e} & ag\bar{e} & -\mathbf{i}fg\bar{e} & \mathbf{i}fg\bar{e} \\ 1 & -1 & cg & -cg & \mathbf{ig} & -\mathbf{ig} & ag\bar{e} & -bg\bar{e} & bg\bar{e} & -ag\bar{e} & -fg\bar{e} & fg\bar{e} \\ 1 & -1 & \mathbf{icg} & -\mathbf{icg} & -\mathbf{ig} & \mathbf{ig} & dg & -dg & -dg & dg & \mathbf{i}fg\bar{e} & -\mathbf{i}fg\bar{e} \\ 1 & -1 & -cg & cg & \mathbf{idg} & \mathbf{idg} & -\mathbf{idg} & dg & dg & -\mathbf{idg} & -dg & -dg \\ 1 & -1 & -cg & cg & -\mathbf{idg} & -\mathbf{idg} & -dg & \mathbf{idg} & \mathbf{idg} & -dg & dg & dg \\ 1 & -1 & -\mathbf{icg} & \mathbf{icg} & -g & g & \mathbf{idg} & -\mathbf{idg} & -\mathbf{idg} & \mathbf{idg} & fg\bar{e} & -fg\bar{e} \end{bmatrix}$$

The four families  $H_{12B}^{(10)}$ ,  $H_{12C}^{(8)}$ ,  $D_{12}^{(7)}$  and  $X_{12}^{(7)}$  contain 101 ACT-equivalence classes. If we include the adjoint, conjugate and transpose of these matrices, we end up with 186 matrices, up to equivalence. This is more than half of all inequivalent BH(4, 12) matrices. The further 19 families containing the remaining BH(4, 12) matrices along with two summarizing tables are available in Appendix A.

**Theorem 6.2.** *Each member of the ACT-equivalence classes of BH(4, 12) matrices can be obtained from 23 partially overlapping infinite affine parametric family of complex Hadamard*

matrices. These matrices, up to ACT-equivalence, can be recognized by the number of  $4 \times 4$  and  $5 \times 5$  vanishing minors they contain.

## 7. THE QUATERNARY COMPLEX HADAMARD MATRICES OF ORDER 14

The exhaustive computer search yielded the following result:

**Theorem 7.1.** *There are precisely 752 BH(4, 14) matrices, up to equivalence, forming 298 ACT-equivalence classes.*

Further analysis revealed 8 matrices that are isolated. For these matrices there exists a neighbourhood of the matrix which does not contain any inequivalent complex Hadamard matrices. We display here one of the 8 known isolated BH(4, 14) matrices:

$$L_{14A}^{(0)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & -i & -1 & -1 & i & -i & -1 & i & 1 \\ 1 & 1 & i & i & i & -i & 1 & -i & -1 & -i & -1 & 1 & -1 & -1 \\ 1 & 1 & i & -i & -i & -i & -1 & i & i & -1 & 1 & i & -i & -1 \\ 1 & 1 & -i & 1 & -1 & i & -1 & -i & 1 & 1 & i & -1 & -1 & -1 \\ 1 & i & -1 & -i & -1 & -1 & -i & i & 1 & -i & -1 & 1 & i & 1 \\ 1 & i & -i & i & 1 & -1 & -1 & -i & -i & -1 & -i & i & 1 & i \\ 1 & -1 & 1 & 1 & -1 & i & i & i & -1 & -i & -i & -i & 1 & -1 \\ 1 & -1 & 1 & -i & i & -1 & 1 & -1 & -i & i & i & i & -i & -i \\ 1 & -1 & i & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -i & -i & i \\ 1 & -1 & -1 & 1 & i & -i & -1 & 1 & -1 & i & 1 & -i & -1 & 1 \\ 1 & -1 & -i & -1 & -i & 1 & i & -i & i & -1 & i & 1 & i & -i \\ 1 & -i & -1 & i & 1 & i & 1 & i & 1 & -1 & -i & -1 & -1 & -i \\ 1 & -i & -1 & -1 & -i & -1 & 1 & 1 & -1 & 1 & i & -1 & 1 & i \end{bmatrix}.$$

The defect  $d(L_{14A}^{(0)}) = 0$  and hence, by Proposition 4.4, the matrix  $L_{14A}^{(0)}$  is isolated amongst all  $14 \times 14$  complex Hadamard matrices. Three additional inequivalent isolated matrices can be obtained by considering the adjoint, conjugate and transpose of  $L_{14A}^{(0)}$ .

**Theorem 7.2.** *There are at least 8 isolated BH(4, 14) matrices, up to equivalence, forming two ACT-equivalence classes.*

As we have found isolated matrices it follows that it is not possible to come up with a universal parametrization scheme for the BH(4,  $n$ ) matrices. This stands in contrast to the real Hadamard matrices which can be parametrized always (cf. Lemma 4.1).

**Theorem 7.3.** *All BH(4, 14) matrices, up to ACT-equivalence, can be recognized by the number of  $4 \times 4$  vanishing minors they contain.*

Although by using Theorem 4.2 we were able to obtain various parametric families starting from many of the BH(4, 14) matrices we constructed, we could not introduce affine parameters into matrices having relatively small defect. Whether or not they are isolated remains an open problem.

For an illustration of the parametrization of BH(4, 14) matrices, we display here a six-parameter family of complex Hadamard matrices:

$$D_{14}^{(6)}(a, b, c, d, e, f) =$$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & ia & -ib & if & ic & -if & -i & -if & -ic & if & ib & -ia & i \\
1 & i\bar{a} & -1 & if & -if & ic\bar{a} & ie\bar{a} & -i & -ie\bar{a} & -ic\bar{a} & -if & if & i & -i\bar{a} \\
1 & -i\bar{b} & i\bar{f} & -1 & id\bar{b} & -i\bar{f} & ie\bar{b} & i & -ie\bar{b} & -i\bar{f} & -id\bar{b} & -i & i\bar{f} & i\bar{b} \\
1 & i\bar{f} & -i\bar{f} & ib\bar{d} & -1 & ic\bar{d} & -ie\bar{d} & i & ie\bar{d} & -ic\bar{d} & -i & -ib\bar{d} & -i\bar{f} & i\bar{f} \\
1 & i\bar{c} & ia\bar{c} & -if & id\bar{c} & -1 & if & -i & if & i & -id\bar{c} & -if & -ia\bar{c} & -i\bar{c} \\
1 & -i\bar{f} & ia\bar{e} & ib\bar{e} & -id\bar{e} & i\bar{f} & -1 & i & -i & i\bar{f} & id\bar{e} & -ib\bar{e} & -ia\bar{e} & -i\bar{f} \\
1 & -i & -i & i & i & -i & i & -1 & i & -i & i & i & -i & -i \\
1 & -i\bar{f} & -ia\bar{e} & -ib\bar{e} & id\bar{e} & i\bar{f} & -i & i & -1 & i\bar{f} & -id\bar{e} & ib\bar{e} & ia\bar{e} & -i\bar{f} \\
1 & -i\bar{c} & -ia\bar{c} & -if & -id\bar{c} & i & if & -i & if & -1 & id\bar{c} & -if & ia\bar{c} & i\bar{c} \\
1 & i\bar{f} & -i\bar{f} & -ib\bar{d} & -i & -ic\bar{d} & ie\bar{d} & i & -ie\bar{d} & ic\bar{d} & -1 & ib\bar{d} & -i\bar{f} & i\bar{f} \\
1 & i\bar{b} & i\bar{f} & -i & -id\bar{b} & -i\bar{f} & -ie\bar{b} & i & ie\bar{b} & -i\bar{f} & id\bar{b} & -1 & i\bar{f} & -i\bar{b} \\
1 & -i\bar{a} & i & if & -if & -ic\bar{a} & -ie\bar{a} & -i & ie\bar{a} & ic\bar{a} & -if & if & -1 & i\bar{a} \\
1 & i & -ia & ib & if & -ic & -if & -i & -if & ic & if & -ib & ia & -1
\end{bmatrix}.$$

The starting-point matrix  $D_{14}^{(6)}(1, 1, 1, 1, 1, 1)$  is a symmetric conference matrix, which was considered along with the five-parameter subfamily  $D_{14}^{(5)}(a, b, c, d, e) = D_{14}^{(6)}(a, b, c, d, e, 1)$  in [15]. The extra parameter was found by the argument outlined in Remark 4.5: we have evaluated the known five-parameter family  $D_{14}^{(6)}(a, b, c, d, e, 1)$  at random fourth roots of unity to obtain some BH(4, 14) matrices, say  $H_1, H_2, \dots, H_{10}$ , and considered the corresponding 10 instances of (10) featuring the same phasing matrix  $R$  in all ten cases. These extra equations heavily constrained the matrix  $R$  resulting in the one-parametric extension we discovered. Despite its large degree of freedom the family  $D_{14}^{(6)}$  contains 14 ACT-classes only, and it seems that a given affine parametric family cannot contain significantly more different ACT-classes in order 14.

For a summarizing table highlighting the main features of BH(4, 14) matrices and for additional examples of parametric families consult a Web site [9]. All results of this work are available in full detail at the Web site which describes all inequivalent BH(4,  $n$ ) matrices, parametric families and summarizing tables for  $10 \leq n \leq 14$ .

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## APPENDIX A. A LIST OF PARAMETRIC FAMILIES OF BH(4, 12) MATRICES

In the following we display the 19 parametric families of order 12 which were not shown in Section 6. These 19 families with the families  $H_{12B}^{(10)}$ ,  $H_{12C}^{(8)}$ ,  $D_{12}^{(7)}$ ,  $X_{12}^{(7)}$  (defined in Section 6) account for all BH(4, 12) matrices, up to ACT-equivalence:

$$L_{12A}^{(1)}(a) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & i & i & -1 & -1 & -1 & -1 & -i & -i \\ 1 & 1 & \bar{a} & -\bar{a} & -1 & -i & ia & i & -i & -ia & i & -1 \\ 1 & 1 & -\bar{a} & \bar{a} & -i & -1 & ia & -i & i & -ia & -1 & i \\ 1 & i & -1 & -i & a & -ia & -ia & ia & -a & -a & ia & a \\ 1 & i & -i & -1 & ia & -a & a & -ia & a & ia & -a & -ia \\ 1 & -1 & a & ia & -ia & a^2 & -ia^2 & -ia^2 & ia^2 & -a^2 & ia^2 & -a \\ 1 & -1 & ia & a & -ia & -a^2 & a^2 & ia^2 & -ia^2 & ia^2 & -ia^2 & -a \\ 1 & -1 & -a & -ia & ia^2 & ia & -a^2 & a^2 & a^2 & -ia^2 & a & -a^2 \\ 1 & -1 & -ia & -a & -ia^2 & ia & ia^2 & -a^2 & -a^2 & a^2 & a & a^2 \\ 1 & -i & i & -1 & ia & -ia & -ia & -a & ia & a & -ia & ia \\ 1 & -i & -1 & i & -a & a & -a & a & -ia & ia & -a & a \end{bmatrix},$$

$$L_{12B}^{(1)}(a) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & a & ia & i & -1 & -1 & -a & -1 & -i & -ia \\ 1 & 1 & 1 & -1 & -1 & -i & 1 & i & -1 & -i & i & -1 \\ 1 & 1 & -1 & ia & a & -ia & -i & -a & -ia & i & -1 & ia \\ 1 & i & -1 & a & -a & -i & -1 & 1 & ia & a & -ia & -a \\ 1 & i & -i & -ia & -ia & ia & -1 & a & -ia & -a & ia & ia \\ 1 & -1 & a & ia^2 & -ia^2 & -ia & ia & -ia & ia & -a & -a & a \\ 1 & -1 & i & -ia & a & -1 & 1 & -i & -a & ia & a & -a \\ 1 & -1 & -a & -ia^2 & ia^2 & a & -ia & -a & ia & -ia & ia & a \\ 1 & -1 & -i & -a & ia & -a & i & ia & -ia & a & -ia & a \\ 1 & -i & i & -a & -a & ia & -i & a & a & i & -1 & -ia \\ 1 & -i & -1 & ia & -ia & i & i & -1 & a & -i & 1 & -a \end{bmatrix},$$

$$\begin{aligned}
B_{12A}^{(4)}(a, b, c, d) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & i & i & -1 & -1 & -1 & -1 & -i & -i \\ 1 & 1 & a\bar{b} & -a\bar{b} & -1 & -i & c\bar{b} & ic\bar{b} & -c\bar{b} & -ic\bar{b} & i & -1 \\ 1 & 1 & -a\bar{b} & a\bar{b} & -1 & -i & -c\bar{b} & -ic\bar{b} & ic\bar{b} & c\bar{b} & -1 & i \\ 1 & 1 & -1 & -1 & 1 & -1 & -ic^2\bar{b}d & ic^2\bar{b}d & -ic\bar{b} & ic\bar{b} & -c\bar{d} & c\bar{d} \\ 1 & 1 & -1 & -1 & -i & i & ic^2\bar{b}d & -ic^2\bar{b}d & c\bar{b} & -c\bar{b} & c\bar{d} & -c\bar{d} \\ 1 & -1 & ib & ib & b & -ib & -c & ic & c & -ic & -ib & -b \\ 1 & -1 & ia & -ia & c & c & -ic & -ic & -c & -c & ic & ic \\ 1 & -1 & ia & -ia & -c & -c & c & c & ic & ic & -ic & -ic \\ 1 & -1 & -ia & ia & ibd\bar{c} & -ibd\bar{c} & ic & -ic & -id & id & ib & -ib \\ 1 & -1 & -ia & ia & -ibd\bar{c} & ibd\bar{c} & c & -c & id & -id & -b & b \\ 1 & -1 & -ib & -ib & -b & ib & -c & ic & -ic & c & b & ib \end{bmatrix}, \\
B_{12B}^{(5)}(a, b, c, d, e) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & i & i & -1 & -1 & -1 & -1 & -i & -i \\ 1 & 1 & i & i & -i & -i & b & ia & -b & -ia & -1 & -1 \\ 1 & 1 & -1 & -1 & a & ib & -ib & -ia & b & -a & ia & -b \\ 1 & 1 & -1 & -1 & -a & -ib & -b & a & ib & ia & -ia & b \\ 1 & 1 & -i & -i & -1 & -1 & ib & -a & -ib & a & i & i \\ 1 & -1 & e & -e & c & -c & -ic & ic & -ic & ic & -c & c \\ 1 & -1 & e & -e & -ic & ic & ic & -ic & ic & -ic & -ic & ic \\ 1 & -1 & ie & -ie & -c & c & id & ad\bar{b} & -id & -ad\bar{b} & ic & -ic \\ 1 & -1 & -e & e & iad\bar{b} & d & -id & -iad\bar{b} & -d & ad\bar{b} & -ad\bar{b} & id \\ 1 & -1 & -e & e & -iad\bar{b} & -d & d & -ad\bar{b} & id & iad\bar{b} & ad\bar{b} & -id \\ 1 & -1 & -ie & ie & ic & -ic & -d & iad\bar{b} & d & -iad\bar{b} & c & -c \end{bmatrix}, \\
B_{12C}^{(5)}(a, b, c, d, e) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & i & i & -1 & -1 & -1 & -1 & -i & -i \\ 1 & 1 & ad\bar{b} & -ad\bar{b} & -1 & -i & a & ia & -a & -ia & i & -1 \\ 1 & 1 & -ad\bar{b} & ad\bar{b} & -1 & -i & -a & -ia & a & ia & i & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & ib\bar{c} & -ib\bar{c} & ib\bar{c} & -ib\bar{c} & -1 & 1 \\ 1 & 1 & -1 & -1 & -i & i & -ib\bar{c} & ib\bar{c} & -ib\bar{c} & ib\bar{c} & -i & i \\ 1 & -1 & e & e & -ie & -e & -ib & -b & ib & b & -e & ie \\ 1 & -1 & d & -d & b & b & -b & -ib & -ib & -b & ib & ib \\ 1 & -1 & d & -d & -b & -b & ib & b & b & ib & -ib & -ib \\ 1 & -1 & -d & d & ic & -ic & b & ib & -ib & -b & -c & c \\ 1 & -1 & -d & d & -ic & ic & ib & b & -b & -ib & c & -c \\ 1 & -1 & -e & -e & ie & e & -ib & -b & ib & b & e & -ie \end{bmatrix}, \\
B_{12D}^{(4)}(a, b, c, d) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & id & -1 & -1 & -1 & -1 & -1 & -id \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & ic\bar{b} & ic\bar{b} & -ic\bar{b} & -ic\bar{b} & -1 \\ 1 & 1 & -1 & 1 & -1 & -id & -1 & c^2\bar{b}^2 & -c^2\bar{b}^2 & c^2\bar{b}^2 & -c^2\bar{b}^2 & id \\ 1 & 1 & -1 & -1 & ic\bar{b} & cd\bar{b} & ic\bar{b} & -ic\bar{b} & -ic\bar{b} & -c^2\bar{b}^2 & c^2\bar{b}^2 & -cd\bar{b} \\ 1 & 1 & -1 & -1 & -ic\bar{b} & -cd\bar{b} & -ic\bar{b} & -c^2\bar{b}^2 & c^2\bar{b}^2 & ic\bar{b} & ic\bar{b} & cd\bar{b} \\ 1 & -1 & ab\bar{c} & ab\bar{c} & -ab\bar{c} & -iabd\bar{c} & -ab\bar{c} & -ac\bar{b} & ac\bar{b} & -ac\bar{b} & ac\bar{b} & iabd\bar{c} \\ 1 & -1 & ab\bar{c} & -ab\bar{c} & ia & -ia & -ia & ac\bar{b} & -ac\bar{b} & ia & ia & -ia \\ 1 & -1 & ab\bar{c} & -ab\bar{c} & -ia & ia & ia & -ia & -ia & ac\bar{b} & -ac\bar{b} & ia \\ 1 & -1 & -ab\bar{c} & ab\bar{c} & -ab\bar{c} & iabd\bar{c} & ab\bar{c} & -ia & ia & ia & -ia & -iabd\bar{c} \\ 1 & -1 & -ab\bar{c} & i\bar{c} & ab\bar{c} & -i\bar{c} & i\bar{c} & ia & \bar{b} & -ia & -\bar{b} & -i\bar{c} \\ 1 & -1 & -ab\bar{c} & -i\bar{c} & ab\bar{c} & i\bar{c} & -i\bar{c} & ia & -\bar{b} & -ia & \bar{b} & i\bar{c} \end{bmatrix}, \\
B_{12E}^{(5)}(a, b, c, d, e) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & ac\bar{b} & 1 & ac\bar{b} & ib\bar{c} & ia & -1 & -1 & -ac\bar{b} & -ac\bar{b} & -ia & -ib\bar{c} \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & ibe & -ibe & -1 & -1 \\ 1 & cd\bar{b} & -1 & cd\bar{b} & -cd\bar{b} & -id & ic^2\bar{d}b^2 & -ic^2\bar{d}b^2 & -icde & icde & id & -cd\bar{b} \\ 1 & cd\bar{b} & -1 & -cd\bar{b} & cd\bar{b} & -id & -ic^2\bar{d}b^2 & ic^2\bar{d}b^2 & -cd\bar{b} & -cd\bar{b} & id & cd\bar{b} \\ 1 & cd\bar{b} & -1 & -cd\bar{b} & -id & id & -cd\bar{b} & -cd\bar{b} & cd\bar{b} & cd\bar{b} & -id & id \\ 1 & -1 & 1 & 1 & -ib\bar{c} & -1 & -ic\bar{b} & ic\bar{b} & -ibe & ibe & -1 & ib\bar{c} \\ 1 & -ac\bar{b} & 1 & -ac\bar{b} & ib\bar{c} & -ia & -1 & -1 & ac\bar{b} & ac\bar{b} & ia & -ib\bar{c} \\ 1 & -1 & 1 & -1 & -ib\bar{c} & 1 & ic\bar{b} & -ic\bar{b} & -1 & -1 & 1 & ib\bar{c} \\ 1 & -cd\bar{b} & -1 & cd\bar{b} & id & -id & -cd\bar{b} & cd\bar{b} & icde & -icde & -id & id \\ 1 & -cd\bar{b} & -1 & i\bar{b} & \bar{c} & id & cd\bar{b} & -i\bar{b} & -e & e & -\bar{c} & -id \\ 1 & -cd\bar{b} & -1 & -i\bar{b} & -\bar{c} & id & cd\bar{b} & i\bar{b} & e & -e & \bar{c} & -id \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
B_{12F}^{(7)}(a, b, c, d, e, f, g) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & f\bar{b} & ig\bar{b} & -1 & -f\bar{b} & -1 & -1 & -1 & -ig\bar{b} \\ 1 & 1 & a\bar{b} & -a\bar{b} & -f\bar{b} & -1 & d\bar{b} & f\bar{b} & ie\bar{b} & -d\bar{b} & -ie\bar{b} & -1 \\ 1 & 1 & -a\bar{b} & a\bar{b} & -1 & -ig\bar{b} & ie\bar{c} & -1 & -ie\bar{c} & ie\bar{c} & -ie\bar{c} & ig\bar{b} \\ 1 & 1 & -1 & -1 & ie\bar{c} & eg\bar{b} & -ie\bar{c} & ie\bar{c} & -ie\bar{b} & -ie\bar{c} & ie\bar{b} & -eg\bar{b} \\ 1 & 1 & -1 & -1 & -ie\bar{c} & -eg\bar{b} & -d\bar{b} & -ie\bar{c} & ie\bar{c} & d\bar{b} & ie\bar{c} & eg\bar{b} \\ 1 & -1 & ib & ib & -ib & g & be\bar{c} & -ib & -be\bar{c} & be\bar{c} & -be\bar{c} & -g \\ 1 & -1 & ia & -ia & ef\bar{c} & -be\bar{c} & -id & -ef\bar{c} & be\bar{c} & id & be\bar{c} & -be\bar{c} \\ 1 & -1 & ia & -ia & -ef\bar{c} & be\bar{c} & -be\bar{c} & ef\bar{c} & e & -be\bar{c} & -e & be\bar{c} \\ 1 & -1 & -ia & ia & ib & -ib & -cd\bar{e} & ib & ic & cd\bar{e} & -ic & -ib \\ 1 & -1 & -ia & ia & -if & -g & cd\bar{e} & if & -ic & -cd\bar{e} & ic & g \\ 1 & -1 & -ib & -ib & if & ib & id & -if & -e & -id & e & ib \end{bmatrix}, \\
B_{12G}^{(7)}(a, b, c, d, e, f, g) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & c\bar{b} & ie\bar{b} & -c\bar{b} & -1 & -1 & -1 & -1 & -ie\bar{b} \\ 1 & 1 & a\bar{b} & -a\bar{b} & -c\bar{b} & -1 & c\bar{b} & if\bar{b} & if\bar{b} & -if\bar{b} & -if\bar{b} & -1 \\ 1 & 1 & -a\bar{b} & a\bar{b} & -1 & -ie\bar{b} & -1 & df\bar{b}^2 & -df\bar{b}^2 & ifg\bar{b}^2 & -ifg\bar{b}^2 & ie\bar{b} \\ 1 & 1 & -1 & -1 & if\bar{b} & ef\bar{b}^2 & if\bar{b} & -if\bar{b} & -if\bar{b} & -ifg\bar{b}^2 & ifg\bar{b}^2 & -ef\bar{b}^2 \\ 1 & 1 & -1 & -1 & -if\bar{b} & -ef\bar{b}^2 & -if\bar{b} & -df\bar{b}^2 & df\bar{b}^2 & if\bar{b} & if\bar{b} & ef\bar{b}^2 \\ 1 & -1 & ib & ib & -ib & e & -ib & -idf\bar{b} & idf\bar{b} & fg\bar{b} & -fg\bar{b} & -e \\ 1 & -1 & ia & -ia & cf\bar{b} & -f & -cf\bar{b} & f & f & -fg\bar{b} & fg\bar{b} & -f \\ 1 & -1 & ia & -ia & -cf\bar{b} & f & cf\bar{b} & idf\bar{b} & -idf\bar{b} & -f & -f & f \\ 1 & -1 & -ia & ia & ib & -ib & ib & -d & d & ig & -ig & -ib \\ 1 & -1 & -ia & ia & -ic & -e & ic & d & -d & -ig & ig & e \\ 1 & -1 & -ib & -ib & ic & ib & -ic & -f & -f & f & f & ib \end{bmatrix}, \\
B_{12H}^{(6)}(a, b, c, d, e, f) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & if & ie & -1 & -1 & -1 & -1 & -ie & -if \\ 1 & 1 & b & -b & -1 & -1 & 1 & 1 & id & -id & -1 & -1 \\ 1 & 1 & -b & b & -1 & -ie & -1 & -1 & 1 & 1 & ie & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & ic & -ic & -id & id & -1 & 1 \\ 1 & 1 & -1 & -1 & -if & 1 & -ic & ic & -1 & -1 & 1 & if \\ 1 & -1 & ia & ia & af & ae & -ac & ac & -ia & -ia & -ae & -af \\ 1 & -1 & iab & -iab & af & -ae & -ia & -ia & ia & ia & ae & -af \\ 1 & -1 & iab & -iab & -af & ae & ac & -ac & ad & -ad & -ae & af \\ 1 & -1 & -iab & iab & ia & -ia & ac & -ac & -ad & ad & -ia & ia \\ 1 & -1 & -iab & iab & -ia & -ae & ia & ia & ad & -ad & ae & -ia \\ 1 & -1 & -ia & -ia & -af & ia & -ac & ac & -ad & ad & ia & af \end{bmatrix}, \\
B_{12I}^{(4)}(a, b, c, d) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & i & i & -1 & -1 & -1 & -1 & -i & -i \\ 1 & 1 & ab & -ab & -1 & -i & bc & ibc & -bc & -ibc & i & -1 \\ 1 & 1 & -ab & ab & -i & -1 & -ibc & -bc & ibc & bc & -1 & i \\ 1 & 1 & -1 & -1 & bc & bc & -bc & -ibc & -ibc & -bc & ibc & ibc \\ 1 & 1 & -1 & -1 & -bc & -bc & ibc & bc & bc & ibc & -ibc & -ibc \\ 1 & -1 & i\bar{b} & i\bar{b} & \bar{b} & -i\bar{b} & -c & c & ic & -ic & -\bar{b} & -i\bar{b} \\ 1 & -1 & ia & -ia & \bar{b}^2 d & -\bar{b}^2 d & -ic & icbd & -icbd & ic & \bar{b} & -\bar{b} \\ 1 & -1 & ia & -ia & -\bar{b}^2 d & \bar{b}^2 d & c & -icbd & icbd & -c & -i\bar{b} & i\bar{b} \\ 1 & -1 & -ia & ia & -\bar{b} & \bar{b} & -bcd & -c & c & bcd & d & -d \\ 1 & -1 & -ia & ia & -i\bar{b} & i\bar{b} & bcd & ic & -ic & -bcd & -d & d \\ 1 & -1 & -i\bar{b} & -i\bar{b} & i\bar{b} & -\bar{b} & ic & -ic & -c & c & i\bar{b} & \bar{b} \end{bmatrix}, \\
B_{12J}^{(4)}(a, b, c, d) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & ic\bar{b} & ib\bar{c} & -1 & -1 & -1 & -1 & -ib\bar{c} & -ic\bar{b} \\ 1 & 1 & ab & -ab & -1 & -ib\bar{c} & bd & ibd & -bd & -ibd & ib\bar{c} & -1 \\ 1 & 1 & -ab & ab & -ic\bar{b} & -1 & -icd & -cd & icd & cd & -1 & ic\bar{b} \\ 1 & 1 & -1 & -1 & cd & bd & -bd & -ibd & -icd & -cd & ibd & icd \\ 1 & 1 & -1 & -1 & -cd & -bd & icd & cd & bd & ibd & -ibd & -icd \\ 1 & -1 & i\bar{b} & i\bar{b} & cb^2 & -i\bar{b} & -cd\bar{b} & icd\bar{b} & cd\bar{b} & -icd\bar{b} & -i\bar{b} & -cb^2 \\ 1 & -1 & ia & -ia & cd\bar{b} & -d & -id & d & -cd\bar{b} & icd\bar{b} & id & -icd\bar{b} \\ 1 & -1 & ia & -ia & -cd\bar{b} & d & cd\bar{b} & -icd\bar{b} & id & -d & -id & icd\bar{b} \\ 1 & -1 & -ia & ia & -cb^2 & \bar{c} & -\bar{b} & \bar{b} & -\bar{b} & \bar{b} & -\bar{c} & cb^2 \\ 1 & -1 & -ia & ia & -i\bar{b} & i\bar{b} & \bar{b} & -\bar{b} & \bar{b} & -\bar{b} & i\bar{b} & -i\bar{b} \\ 1 & -1 & -i\bar{b} & -i\bar{b} & i\bar{b} & -\bar{c} & id & -d & -id & d & \bar{c} & i\bar{b} \end{bmatrix},
\end{aligned}$$



$$B_{12K}^{(4)}(a, b, c, d) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & i\bar{b}c & ibc & -1 & -1 & -1 & -1 & -ibc & -i\bar{b}c \\ 1 & 1 & ab & -ab & -1 & -ibc & bd & ibd & -bd & -ibd & ibc & -1 \\ 1 & 1 & -ab & ab & -i\bar{b}c & -1 & -ib^2cd & -b^2cd & ib^2cd & b^2cd & -1 & i\bar{b}c \\ 1 & 1 & -1 & -1 & bd & -b^2cd & -bd & b^2cd & -ib^2cd & ibd & ib^2cd & -ibd \\ 1 & 1 & -1 & -1 & -bd & b^2cd & ib^2cd & -ibd & bd & -b^2cd & -ib^2cd & ibd \\ 1 & -1 & i\bar{b} & i\bar{b} & \bar{b}^2c & -i\bar{b} & -bcd & ibcd & bcd & -ibcd & -i\bar{b} & -\bar{b}^2c \\ 1 & -1 & ia & -ia & d & bcd & -id & -ibcd & -bcd & -d & ibcd & id \\ 1 & -1 & ia & -ia & -d & -bcd & bcd & d & id & ibcd & -ibcd & -id \\ 1 & -1 & -ia & ia & -\bar{b}^2c & c & -\bar{b} & \bar{b} & -\bar{b} & \bar{b} & -c & \bar{b}^2c \\ 1 & -1 & -ia & ia & -i\bar{b} & i\bar{b} & \bar{b} & -\bar{b} & \bar{b} & -\bar{b} & i\bar{b} & -i\bar{b} \\ 1 & -1 & -i\bar{b} & -i\bar{b} & i\bar{b} & -c & id & -d & -id & d & c & i\bar{b} \end{bmatrix},$$

$$B_{12L}^{(5)}(a, b, c, d, e) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & i\bar{b}\bar{c} & ic\bar{b} & -1 & -1 & -1 & -1 & -ic\bar{b} & -i\bar{b}\bar{c} \\ 1 & 1 & a\bar{b} & -a\bar{b} & -1 & -ic\bar{b} & d\bar{c} & id\bar{b} & -d\bar{c} & -id\bar{b} & ic\bar{b} & -1 \\ 1 & 1 & -a\bar{b} & a\bar{b} & -i\bar{b}\bar{c} & -1 & ie\bar{b} & ceb^2 & -ie\bar{b} & -ceb^2 & -1 & i\bar{b}\bar{c} \\ 1 & 1 & -1 & -1 & d\bar{c} & -ceb^2 & -d\bar{c} & -id\bar{b} & ie\bar{b} & ceb^2 & id\bar{b} & -ie\bar{b} \\ 1 & 1 & -1 & -1 & -d\bar{c} & ceb^2 & -ie\bar{b} & -ceb^2 & d\bar{c} & id\bar{b} & -id\bar{b} & ie\bar{b} \\ 1 & -1 & ib & ib & b^2\bar{c} & -ib & e & -ice\bar{b} & -e & ice\bar{b} & -ib & -b^2\bar{c} \\ 1 & -1 & ia & -ia & ibd\bar{c} & ice\bar{b} & -ibd\bar{c} & d & e & -ice\bar{b} & -d & -e \\ 1 & -1 & ia & -ia & -ibd\bar{c} & -ice\bar{b} & -e & ice\bar{b} & ibd\bar{c} & -d & d & e \\ 1 & -1 & -ia & ia & -b^2\bar{c} & c & ib & -ib & ib & -ib & -c & b^2\bar{c} \\ 1 & -1 & -ia & ia & -ib & ib & -ib & ib & -ib & ib & ib & -ib \\ 1 & -1 & -ib & -ib & ib & -c & ibd\bar{c} & -d & -ibd\bar{c} & d & c & ib \end{bmatrix},$$

$$B_{12M}^{(3)}(a, b, c) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & ab & bc & iab & ibc & -bc & -1 & -1 & -ab & -ibc & -iab \\ 1 & 1 & ab & -bc & -ab & -ibc & bc & i & -1 & -i & ibc & -1 \\ 1 & 1 & -ab & i & ab & -ibc & ibc & -i & -i & -1 & -1 & i \\ 1 & 1 & -ab & -1 & -iab & ibc & -ibc & -1 & i & ab & -i & iab \\ 1 & 1 & -1 & -i & -1 & -1 & -1 & 1 & 1 & i & i & -i \\ 1 & -1 & i\bar{b} & \bar{b} & i\bar{b} & -i\bar{b} & -i\bar{b} & -i\bar{b} & i\bar{b} & \bar{b} & -\bar{b} & -\bar{b} \\ 1 & -1 & ia & -\bar{b} & a & -c & c & i\bar{b} & \bar{b} & -a & -i\bar{b} & -ia \\ 1 & -1 & ia & -i\bar{b} & -ia & c & -c & \bar{b} & -\bar{b} & -\bar{b} & \bar{b} & i\bar{b} \\ 1 & -1 & -ia & c & -a & -c & ic & i\bar{b} & -i\bar{b} & a & -ic & ia \\ 1 & -1 & -i\bar{b} & i\bar{b} & -i\bar{b} & i\bar{b} & i\bar{b} & -i\bar{b} & i\bar{b} & -i\bar{b} & i\bar{b} & -i\bar{b} \\ 1 & -1 & -ia & -c & ia & c & -ic & -\bar{b} & -i\bar{b} & i\bar{b} & ic & \bar{b} \end{bmatrix},$$

$$B_{12N}^{(6)}(a, b, c, d, e, f) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & ab & bc & i & i & -bc & -1 & -1 & -ab & -i & -i \\ 1 & 1 & iab & i & -i & -i & i & ibf & -ibf & -iab & -1 & -1 \\ 1 & 1 & -1 & -bc & bd & -bd & bc & -ibf & ibf & -1 & be & -be \\ 1 & 1 & -ab & -i & -1 & -1 & -i & i & i & ab & -be & be \\ 1 & 1 & -iab & -1 & -bd & bd & -1 & -i & -i & iab & i & i \\ 1 & -1 & \bar{b} & ic & id & -id & -ic & -if & if & \bar{b} & -\bar{b} & -\bar{b} \\ 1 & -1 & a & -\bar{b} & -i\bar{b} & -i\bar{b} & -\bar{b} & i\bar{b} & i\bar{b} & -a & \bar{b} & \bar{b} \\ 1 & -1 & i\bar{b} & -ic & d & -d & ic & -i\bar{b} & -i\bar{b} & i\bar{b} & -e & e \\ 1 & -1 & -\bar{b} & \bar{b} & -d & d & \bar{b} & -f & f & -\bar{b} & ie & -ie \\ 1 & -1 & -a & c & -id & id & -c & f & -f & a & e & -e \\ 1 & -1 & -i\bar{b} & -c & i\bar{b} & i\bar{b} & c & if & -if & -i\bar{b} & -ie & ie \end{bmatrix},$$

$$B_{12O}^{(3)}(a, b, c) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \bar{a}b & ab & iab & i\bar{a}b & -ab & -1 & -\bar{a}b & -1 & -iab & -i\bar{a}b \\ 1 & 1 & i & iab & -iab & -i & ab & iac & -1 & -iac & -ab & -1 \\ 1 & 1 & -1 & -ab & ab & -1 & -iab & -iac & i & iac & iab & -i \\ 1 & 1 & -\bar{a}b & -iab & -1 & i & iab & -i & \bar{a}b & -i & -1 & i \\ 1 & 1 & -i & -1 & -ab & -i\bar{a}b & -1 & i & -i & i & ab & i\bar{a}b \\ 1 & -1 & b & -b & ab^2 & -\bar{a} & -b & ic & b & -ic & -ab^2 & \bar{a} \\ 1 & -1 & \bar{a} & -ab^2 & -ib & ib & ab^2 & -ic & -\bar{a} & ic & -ib & ib \\ 1 & -1 & ib & -iab^2 & -ab^2 & b & -ab^2 & ab^2 & -b & ab^2 & iab^2 & -ib \\ 1 & -1 & -b & ab^2 & -iab^2 & -ib & iab^2 & -ab^2 & ib & -ab^2 & ab^2 & b \\ 1 & -1 & -\bar{a} & iab^2 & iab^2 & \bar{a} & -iab^2 & c & \bar{a} & -c & -iab^2 & -\bar{a} \\ 1 & -1 & -ib & b & ib & -b & b & -c & -ib & c & ib & -b \end{bmatrix},$$

$$B_{12P}^{(4)}(a, b, c, d) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & cd & ac & iac & icd & -ac & -1 & -cd & -1 & -icd & -iac \\ 1 & 1 & i & iac & -iac & -i & ac & ibc & -1 & -ibc & -1 & -ac \\ 1 & 1 & -cd & -1 & i & -icd & -i & -ibc & icd & ibc & cd & -1 \\ 1 & 1 & -1 & -iac & -1 & i & i & -i & cd & -i & -cd & iac \\ 1 & 1 & -i & -ac & -i & -1 & -1 & i & -icd & i & icd & ac \\ 1 & -1 & \bar{c} & -\bar{c} & -\bar{c} & \bar{c} & -\bar{c} & ib & \bar{c} & -ib & \bar{c} & -\bar{c} \\ 1 & -1 & d & -a & -ia & id & a & -ib & -d & ib & -id & ia \\ 1 & -1 & i\bar{c} & -ia & ia & -i\bar{c} & -a & \bar{c} & -\bar{c} & \bar{c} & -\bar{c} & a \\ 1 & -1 & -d & \bar{c} & -i\bar{c} & -id & i\bar{c} & -\bar{c} & id & -\bar{c} & d & \bar{c} \\ 1 & -1 & -\bar{c} & ia & \bar{c} & i\bar{c} & -i\bar{c} & b & d & -b & -d & -ia \\ 1 & -1 & -i\bar{c} & a & i\bar{c} & -\bar{c} & \bar{c} & -b & -id & b & id & -a \end{bmatrix},$$

$$B_{12Q}^{(4)}(a, b, c, d) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & b & b & b & b & -b & -b & -b & -1 & -1 & -b \\ 1 & 1 & ib & -b & -b & -ib & b & b & ib & -1 & -1 & -ib \\ 1 & 1 & -1 & i & -i & -1 & i & -i & -1 & 1 & 1 & -1 \\ 1 & 1 & -b & -i & -1 & ib & -i & -1 & -ib & i & i & b \\ 1 & 1 & -ib & -1 & i & -b & -1 & i & b & -i & -i & ib \\ 1 & -1 & a & ic & -a & -ic & ia & -c & -ia & d & -d & c \\ 1 & -1 & c & -c & -ic & ic & -ic & c & -c & -id & id & ic \\ 1 & -1 & ia & c & -ia & -c & -a & ic & a & -d & d & -ic \\ 1 & -1 & -a & a & ia & -ia & -ia & a & -a & id & -id & ia \\ 1 & -1 & -c & -ic & ic & c & ic & -ic & c & -id & id & -c \\ 1 & -1 & -ia & -a & a & ia & a & -a & ia & id & -id & -ia \end{bmatrix}.$$

Table 2 summarizes the relations between BH(4, 12) matrices and parametric families by showing which ACT-classes are included in each of the 23 parametric families. An ACT-class number is underlined if the ACT-class belongs to only one of the parametric families listed in the table.

Table 3 summarizes the properties of BH(4, 12) matrices.

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TABLE 2. Summary of the parametric families of order 12

Family	ACT-classes
$H_{12B}^{(10)}$	1, 2, 3, 4, 5, 6, 7, 9, 10, 11, <u>12</u> , 13, 14, 15, 16, 17, <u>18</u> , 19, 21, 22, 23, 28, <u>29</u> , <u>45</u> , <u>46</u> , 47, <u>48</u> , <u>49</u> , 59, <u>62</u> , <u>65</u> , <u>66</u> , <u>70</u> , 78, <u>79</u> , 81
$H_{12C}^{(8)}$	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 15, 16, 17, 19, <u>20</u> , 21, 22, 23, 24, 25, 26, <u>27</u> , 28, 32, 33, 34, 37, 38, 39, 40, <u>50</u> , 51, <u>52</u> , <u>53</u> , <u>55</u> , <u>57</u> , 58, 60, 61, 63, 64
$D_{12}^{(7)}$	103, 104, 105, 106, 107, 108, 109, <u>110</u> , 111, 112, <u>113</u> , <u>114</u> , <u>115</u> , 117, 127, <u>128</u> , 129, 130, <u>131</u> , 138
$X_{12}^{(7)}$	<u>44</u> , 75, <u>76</u> , 107, 116, 118, <u>120</u> , 121, 122, 123, 124, 125, 126, 133, 134, <u>135</u> , <u>136</u> , <u>137</u> , 139, 140, 154, <u>160</u>
$L_{12A}^{(1)}$	<u>164</u> , <u>165</u>
$L_{12B}^{(1)}$	<u>166</u> , <u>167</u>
$B_{12A}^{(4)}$	30, 31, 68, <u>69</u> , 74, 87, <u>88</u> , 156, 157
$B_{12B}^{(5)}$	103, 104, 105, 106, 107, 108, 116, 117, 121, 122, 123, 124, 125, 126, 127, 129, 130, <u>132</u> , 133, 134, 141, 142, 143, 144, <u>153</u> , <u>158</u> , <u>161</u> , <u>162</u> , <u>163</u>
$B_{12C}^{(5)}$	32, 33, 35, 36, 37, 38, 58, 60, 61, 63, 64, <u>71</u> , <u>72</u> , <u>80</u>
$B_{12D}^{(4)}$	1, 2, 3, 6, 10, 21, 24, 41, <u>56</u> , 73
$B_{12E}^{(5)}$	1, 2, 3, 4, 6, 10, 15, 21, 24, 26, 39, 40, 41, <u>42</u> , 51, <u>67</u>
$B_{12F}^{(7)}$	1, 2, 3, 8, 21, 22, 23, 24, 25, 37, 38, 74, <u>83</u> , 93, <u>97</u> , 111, 138
$B_{12G}^{(7)}$	1, 2, 3, 8, 9, 21, 22, 23, 24, 25, 34, 37, 38, 73, <u>84</u> , <u>85</u> , <u>94</u> , 139, 140
$B_{12H}^{(6)}$	1, 2, 3, 8, 9, 34, <u>86</u> , 147
$B_{12I}^{(4)}$	30, 31, 89, <u>91</u> , 148, <u>150</u>
$B_{12J}^{(4)}$	13, 35, 47, 59, 78, 81, <u>92</u> , <u>95</u> , 105, 121
$B_{12K}^{(4)}$	13, 36, 47, 59, 78, 81, <u>90</u> , <u>96</u> , 108, 123
$B_{12L}^{(5)}$	1, 2, 3, 4, 21, 22, 23, 26, 32, 33, 58, <u>98</u> , <u>99</u> , <u>100</u> , <u>101</u> , 141, 142, 143, 144
$B_{12M}^{(3)}$	30, 43, 87, 89, <u>102</u> , 118, 119, 133, 145
$B_{12N}^{(6)}$	119, 145, <u>146</u> , 147, 154
$B_{12O}^{(3)}$	148, <u>149</u>
$B_{12P}^{(4)}$	74, 75, 93, 107, 109, 111, 112, 122, <u>151</u> , <u>152</u> , 156, 157
$B_{12Q}^{(4)}$	30, 43, <u>54</u> , 68, <u>77</u> , <u>82</u> , 118, 119, <u>155</u> , <u>159</u>

TABLE 3. Summary of the BH(4, 12) matrices

ACT class	Equiv. classes	Family, coordinates	ACT	HBS	Auto. order	Defect	Orbit	$\mathbb{Z}_4$ rank	Invariant ( $4 \times 4, 5 \times 5$ )
1	1	$H_{12B}^{(10)}(1, -1, -1, -1, 1, -1, 1, 1, -1, 1)$	YYY	YNY	380160	55	10	10	(109890, 205920)
2	2, 8	$H_{12B}^{(10)}(-1, -1, 1, -1, -1, -1, -1, -1, -i, 1, 1)$	NYN	NNN	5760	45	10	9	(86130, 146400)
3	3, 23	$H_{12B}^{(10)}(-1, -1, 1, -i, 1, 1, -1, -1, -i, 1)$	NYN	NNN	256	35	10	8	(66578, 106208)
4	4	$H_{12B}^{(10)}(i, -1, -i, 1, -1, 1, 1, -i, 1, -i)$	YYY	YNY	192	31	10	8	(55554, 86304)
5	5, 13	$H_{12B}^{(10)}(1, -i, -1, -i, -1, 1, 1, -i, -1, 1)$	NYN	NNN	64	29	10	7	(56258, 82848)
6	6	$H_{12B}^{(10)}(i, 1, -1, -i, -1, -1, -1, -1, i, i)$	YYY	YNY	192	37	10	8	(68082, 105392)
7	7, 30	$H_{12B}^{(10)}(i, -1, i, -i, 1, -1, i, i, -1, -1)$	NYN	NNN	192	25	10	7	(52722, 85920)
8	9, 47	$H_{12C}^{(10)}(i, i, 1, i, -1, -1, -1, -1)$	NYN	NNN	48	25	8	7	(50238, 78720)
9	10, 66	$H_{12B}^{(10)}(i, 1, 1, i, -i, -1, 1, -1, -i, 1)$	NYN	NNN	96	25	10	7	(50442, 77328)
10	11, 20	$H_{12B}^{(10)}(1, i, -i, 1, -i, -1, -1, i, 1, 1)$	NYN	NNN	32	29	10	7	(52114, 75536)
11	12, 21	$H_{12B}^{(10)}(-1, -i, 1, i, -1, i, -i, 1, -1, 1)$	NYN	NNN	128	31	10	7	(55346, 79488)
12	14	$H_{12B}^{(10)}(1, 1, -1, i, -1, 1, i, -i, i, -1)$	YYY	YNY	64	27	10	6	(51010, 70016)
13	15	$H_{12B}^{(10)}(-1, 1, 1, 1, -i, -1, -1, i, -1, i)$	YYY	YNY	256	25	10	6	(53218, 74144)
14	16, 89	$H_{12B}^{(10)}(i, -i, -1, i, -i, 1, -1, i, i, i)$	NYN	NNN	32	21	10	6	(43122, 64976)
15	17, 29	$H_{12B}^{(10)}(-1, 1, 1, i, -i, 1, i, i, i, i)$	NYN	NNN	32	23	10	7	(43642, 65648)
16	18, 59	$H_{12B}^{(10)}(i, 1, -1, i, -i, i, -i, 1, 1, i)$	NYN	NNN	16	23	10	6	(43862, 60696)
17	19, 79	$H_{12B}^{(10)}(i, i, -1, -i, -i, 1, 1, -1, -i, 1)$	NYN	NNN	64	25	10	6	(45578, 61584)
18	22	$H_{12B}^{(10)}(-1, -i, 1, -1, -1, -1, 1, -i, -i, -i)$	YYY	NNY	64	25	10	6	(48738, 71904)
19	24, 61	$H_{12B}^{(10)}(1, -1, 1, i, -i, -1, 1, -1, -1, i)$	NYN	NNN	16	23	10	6	(44178, 60440)
20	25, 46	$H_{12C}^{(8)}(i, -1, -1, -i, 1, 1, -i, i)$	NYN	NNN	8	21	8	6	(41310, 59236)
21	26	$H_{12B}^{(10)}(i, -i, -i, -i, i, -1, -1, -i, i, 1)$	YYY	YNY	128	27	10	8	(47442, 74080)
22	27, 57	$H_{12B}^{(10)}(i, -i, 1, i, i, i, 1, i, -i, 1)$	NYN	NNN	64	21	10	7	(44338, 60768)
23	28	$H_{12B}^{(10)}(-1, -1, -1, 1, 1, -1, i, i, i, i)$	YYY	YNY	768	31	10	8	(55842, 78624)
24	31, 50	$H_{12C}^{(8)}(i, i, i, 1, -i, i, -i, -i)$	NYN	NNN	16	19	8	7	(36550, 53888)
25	32, 86	$H_{12C}^{(8)}(1, i, -i, -1, -i, -i, i, -i)$	NYN	NNN	16	15	8	6	(34900, 47240)
26	33, 48	$H_{12C}^{(8)}(i, -i, 1, i, 1, i, -1, i)$	NYN	NNN	24	21	8	7	(39186, 58512)
27	34, 52	$H_{12C}^{(8)}(i, -i, -i, i, 1, -1, 1, 1)$	NYN	NNN	24	17	8	6	(39198, 62684)
28	35, 100	$H_{12B}^{(10)}(i, i, -i, -1, -1, -1, 1, 1, -i, 1)$	NYN	NNN	64	17	10	6	(41306, 67376)
29	36	$H_{12B}^{(10)}(1, i, -1, i, -1, 1, -i, i, -i, i)$	YYY	YNY	768	17	10	6	(44802, 81440)
30	37, 174	$B_{12A}^{(4)}(i, -i, 1, -i)$	NYN	NNN	16	17	4	7	(26066, 41568)
31	38, 126	$B_{12A}^{(4)}(i, i, 1, 1)$	NYN	NNN	16	15	4	7	(25814, 35804)
32	39, 125	$H_{12C}^{(8)}(i, -1, -i, -i, 1, -i, -i, -i)$	NYN	NNN	64	17	8	6	(31838, 42304)
33	40, 228	$H_{12C}^{(8)}(i, -i, i, -i, -i, 1, -1, i)$	NYN	NNN	64	21	8	6	(32094, 47296)
34	41, 81	$H_{12C}^{(8)}(1, -i, i, 1, i, -1, -1, -i)$	NYN	NNN	16	15	8	6	(37760, 56304)
35	42, 230	$B_{12C}^{(5)}(-1, i, -i, -1, -i)$	NYN	NNN	64	15	5	7	(25846, 39904)
36	43, 231	$B_{12C}^{(5)}(-1, -1, 1, 1, -1)$	NYN	NNN	64	17	5	7	(26422, 41216)
37	44, 96	$H_{12C}^{(8)}(i, -i, -i, -i, i, i, i, -i)$	NYN	NNN	64	21	8	6	(34230, 51632)
38	45, 94	$H_{12C}^{(8)}(i, -1, i, -i, 1, -1, -1, i)$	NYN	NNN	64	17	8	6	(37542, 62656)
39	49, 71	$H_{12C}^{(8)}(-1, 1, -i, i, -i, -i, i, i)$	NYN	NNN	8	15	8	6	(32374, 46084)
40	51, 101	$H_{12C}^{(8)}(i, i, -1, -i, -i, i, 1, i)$	NYN	NNN	8	15	8	6	(32738, 46548)
41	53	$B_{12D}^{(4)}(1, 1, 1, 1)$	YYY	YNY	4	15	4	7	(29004, 39852)
42	54	$B_{12E}^{(5)}(-1, -1, 1, -1, 1)$	YYY	YNY	12	13	5	7	(29292, 40946)
43	55, 194	$B_{12M}^{(3)}(1, i, 1)$	NYN	NNN	4	9	3	7	(23960, 38492)
44	56, 234	$X_{12}^{(7)}(1, -i, -i, 1, i, i, 1)$	NYN	NNN	24	15	7	7	(29100, 50052)
45	58	$H_{12B}^{(10)}(1, 1, 1, 1, i, -i, -i, 1, 1, -i)$	YYY	YNY	64	15	10	6	(39058, 47904)
46	60, 64	$H_{12B}^{(10)}(1, i, 1, -i, i, -i, -1, 1, i, 1)$	NYN	NNN	32	19	10	5	(39594, 51472)
47	62, 63	$H_{12B}^{(10)}(i, i, -i, -i, -i, -i, -i, i, 1, i)$	NYN	NNN	64	19	10	5	(40850, 51776)
48	65, 73	$H_{12B}^{(10)}(i, 1, 1, -i, i, -1, 1, i, -i, i)$	NYN	NNN	32	21	10	5	(41066, 58512)
49	67, 80	$H_{12B}^{(10)}(-1, 1, -i, i, -1, i, -i, -1, -1, -1)$	NYN	NNN	96	25	10	5	(44178, 55200)
50	68, 84	$H_{12C}^{(8)}(i, -1, 1, i, -i, i, i, -1)$	NYN	NNN	8	17	8	5	(33462, 45720)
51	69	$H_{12C}^{(8)}(i, 1, -1, 1, -1, -i, -i, -i)$	YYY	YNY	16	23	8	6	(39598, 52416)
52	70, 90	$H_{12C}^{(8)}(i, -i, i, -1, 1, i, 1, -i)$	NYN	NNN	8	13	8	5	(32730, 48504)
53	72, 77	$H_{12C}^{(8)}(i, 1, i, 1, 1, -i, -i, -i)$	NYN	NNN	16	19	8	5	(34626, 43160)
54	74, 233	$B_{12Q}^{(4)}(1, i, i, 1)$	NYN	NNN	8	9	4	6	(22330, 33236)
55	75, 129	$H_{12C}^{(8)}(i, i, i, 1, i, -1, 1, i)$	NYN	NNN	16	13	8	5	(32282, 48528)

ACT class	Equiv. classes	Family, coordinates	ACT	HBS	Auto. order	Defect	Orbit	$\mathbb{Z}_4$ rank	Invariant ( $4 \times 4, 5 \times 5$ )
56	76, 112	$B_{12D}^{(4)}(\mathbf{i}, \mathbf{i}, \mathbf{i}, 1)$	NYN	NNN	8	15	4	6	(24670, 28992)
57	78, 82	$H_{12C}^{(8)}(1, \mathbf{i}, \mathbf{i}, -\mathbf{i}, -\mathbf{i}, \mathbf{i}, 1, 1)$	NYN	NNN	8	15	8	5	(34136, 45692)
58	83, 121	$H_{12C}^{(8)}(\mathbf{i}, \mathbf{i}, -\mathbf{i}, \mathbf{i}, 1, \mathbf{i}, 1, -\mathbf{i})$	NYN	NNN	16	15	8	6	(29822, 44900)
59	85, 91	$H_{12B}^{(10)}(\mathbf{i}, \mathbf{i}, -\mathbf{i}, -\mathbf{i}, 1, -1, -\mathbf{i}, -\mathbf{i}, \mathbf{i}, 1)$	NYN	NNN	128	23	10	5	(41618, 56864)
60	87, 128	$H_{12C}^{(8)}(\mathbf{i}, -1, \mathbf{i}, -\mathbf{i}, -\mathbf{i}, 1, -1, -\mathbf{i})$	NYN	NNN	32	13	8	5	(32006, 50144)
61	88, 124	$H_{12C}^{(8)}(\mathbf{i}, -1, \mathbf{i}, \mathbf{i}, -\mathbf{i}, -\mathbf{i}, \mathbf{i}, -\mathbf{i})$	NYN	NNN	32	13	8	5	(29470, 41112)
62	92, 98	$H_{12B}^{(10)}(-1, \mathbf{i}, -\mathbf{i}, \mathbf{i}, \mathbf{i}, -1, 1, -\mathbf{i}, -\mathbf{i}, 1)$	NYN	NNN	32	19	10	5	(38602, 55664)
63	93, 110	$H_{12C}^{(8)}(\mathbf{i}, -1, -\mathbf{i}, -\mathbf{i}, 1, \mathbf{i}, -1, -\mathbf{i})$	NYN	NNN	32	15	8	5	(31710, 50448)
64	95, 109	$H_{12C}^{(8)}(\mathbf{i}, \mathbf{i}, -\mathbf{i}, \mathbf{i}, \mathbf{i}, -1, -\mathbf{i}, -\mathbf{i})$	NYN	NNN	32	19	8	5	(29734, 43656)
65	97	$H_{12B}^{(10)}(1, \mathbf{i}, -1, -\mathbf{i}, -\mathbf{i}, -1, -1, 1, -\mathbf{i}, \mathbf{i})$	YYY	YNY	64	21	10	6	(37842, 57248)
66	99	$H_{12B}^{(10)}(\mathbf{i}, -1, \mathbf{i}, 1, -1, -\mathbf{i}, -\mathbf{i}, -1, 1, -\mathbf{i})$	YYY	YNN	64	17	10	6	(36386, 52960)
67	102, 103	$B_{12E}^{(5)}(\mathbf{i}, 1, -1, -1, -1)$	NYN	NNN	4	9	5	6	(24798, 33548)
68	104, 232	$B_{12A}^{(4)}(-1, -\mathbf{i}, -\mathbf{i}, 1)$	NYN	NNN	8	13	4	6	(22642, 39868)
69	105, 123	$B_{12A}^{(4)}(1, 1, 1, -\mathbf{i})$	NYN	NNN	8	11	4	6	(23008, 36284)
70	106, 130	$H_{12B}^{(10)}(\mathbf{i}, -\mathbf{i}, 1, -\mathbf{i}, \mathbf{i}, -1, \mathbf{i}, \mathbf{i}, -\mathbf{i}, \mathbf{i})$	NYN	NNN	128	17	10	5	(36946, 63136)
71	107, 138	$B_{12C}^{(5)}(-1, -1, -1, -1, \mathbf{i})$	NYN	NNN	32	11	5	6	(21982, 32816)
72	108, 140	$B_{12C}^{(5)}(-1, -1, 1, 1, -\mathbf{i})$	NYN	NNN	32	11	5	6	(21790, 32720)
73	111, 134	$B_{12D}^{(4)}(\mathbf{i}, 1, 1, 1)$	NYN	NNN	16	15	4	6	(27956, 36400)
74	113, 131	$B_{12A}^{(4)}(\mathbf{i}, -1, 1, \mathbf{i})$	NYN	NNN	16	15	4	6	(28158, 45072)
75	114, 177	$X_{12}^{(7)}(\mathbf{i}, -\mathbf{i}, -\mathbf{i}, -\mathbf{i}, -1, 1, -1)$	NYN	NNN	16	13	7	6	(27362, 49384)
76	115, 180	$X_{12}^{(7)}(-1, 1, 1, 1, \mathbf{i}, \mathbf{i}, \mathbf{i})$	NYN	NNN	16	15	7	6	(27038, 43736)
77	116, 243	$B_{12Q}^{(4)}(1, 1, \mathbf{i}, 1)$	NYN	NNN	8	13	4	6	(22292, 34064)
78	117	$H_{12B}^{(10)}(1, -\mathbf{i}, -1, -\mathbf{i}, -\mathbf{i}, -1, -\mathbf{i}, -1, 1, \mathbf{i})$	YYY	NNY	128	17	10	6	(34834, 47456)
79	118	$H_{12B}^{(10)}(1, -\mathbf{i}, -1, \mathbf{i}, -\mathbf{i}, \mathbf{i}, 1, \mathbf{i}, -\mathbf{i}, 1)$	YYY	YNY	192	13	10	6	(35058, 42528)
80	119, 120	$B_{12C}^{(5)}(\mathbf{i}, -1, 1, \mathbf{i}, -1)$	NYN	NNN	32	15	5	7	(20406, 28936)
81	122, 229	$H_{12B}^{(10)}(\mathbf{i}, -\mathbf{i}, \mathbf{i}, -\mathbf{i}, 1, 1, -1, 1, \mathbf{i}, -\mathbf{i})$	NYN	NNN	768	25	10	6	(36018, 55008)
82	127, 209	$B_{12Q}^{(4)}(1, \mathbf{i}, \mathbf{i}, \mathbf{i})$	NYN	NNN	16	13	4	7	(20360, 31464)
83	132, 133, 263, 265	$B_{12F}^{(7)}(-1, -\mathbf{i}, -1, 1, 1, \mathbf{i}, -\mathbf{i})$	NNN	NNN	48	7	7	5	(26844, 38568)
84	135, 264	$B_{12G}^{(7)}(-1, -\mathbf{i}, -1, 1, 1, -\mathbf{i}, -1)$	NYN	NNN	24	9	7	5	(26778, 35196)
85	136, 271	$B_{12G}^{(7)}(\mathbf{i}, -1, \mathbf{i}, -1, \mathbf{i}, \mathbf{i}, -\mathbf{i})$	NYN	NNN	32	11	7	5	(30406, 44432)
86	137, 272	$B_{12H}^{(6)}(-1, -\mathbf{i}, \mathbf{i}, -1, 1, 1)$	NYN	NNN	40	9	6	5	(30420, 40300)
87	139, 310	$B_{12A}^{(4)}(-1, \mathbf{i}, \mathbf{i}, -\mathbf{i})$	NYN	NYN	8	9	4	7	(18514, 31824)
88	141, 277	$B_{12A}^{(4)}(1, -\mathbf{i}, -1, \mathbf{i})$	NYN	NNN	8	7	4	7	(18432, 27188)
89	142, 309	$B_{12I}^{(4)}(1, 1, 1, \mathbf{i})$	NYN	NYN	8	9	4	7	(18600, 31816)
90	143, 295	$B_{12K}^{(4)}(1, 1, -1, 1)$	NYN	NYN	32	11	4	6	(21246, 31936)
91	144, 146, 278, 279	$B_{12I}^{(4)}(1, 1, 1, 1)$	NNN	NYN	16	9	4	7	(18806, 29520)
92	145, 288	$B_{12J}^{(4)}(1, 1, 1, 1)$	NYN	NYN	32	13	4	6	(21494, 45632)
93	147, 268	$B_{12F}^{(7)}(1, -\mathbf{i}, 1, \mathbf{i}, 1, 1, \mathbf{i})$	NYN	NYN	32	17	7	5	(28422, 52888)
94	148, 270	$B_{12G}^{(7)}(\mathbf{i}, -1, \mathbf{i}, -1, 1, 1, 1)$	NYN	NNN	32	11	7	5	(27878, 34616)
95	149, 294	$B_{12J}^{(4)}(1, 1, -1, 1)$	NYN	NNN	32	11	4	6	(20998, 31464)
96	150, 291	$B_{12K}^{(4)}(1, 1, 1, 1)$	NYN	NYN	32	13	4	6	(21566, 46832)
97	151, 273	$B_{12F}^{(7)}(\mathbf{i}, -1, \mathbf{i}, -\mathbf{i}, -1, \mathbf{i}, -\mathbf{i})$	NYN	NYN	160	17	7	5	(30350, 60480)
98	152, 293	$B_{12L}^{(5)}(-1, -1, -1, \mathbf{i}, -\mathbf{i})$	NYN	NYN	32	11	5	6	(21118, 36288)
99	153, 275	$B_{12L}^{(5)}(1, 1, -1, 1, -1)$	NYN	NYN	32	9	5	6	(21478, 29888)
100	154, 276	$B_{12L}^{(5)}(1, 1, 1, 1, 1)$	NYN	NNN	32	9	5	6	(21070, 28800)
101	155, 292	$B_{12L}^{(5)}(1, 1, -1, -\mathbf{i}, -\mathbf{i})$	NYN	NNN	32	13	5	6	(21118, 36864)
102	156, 312	$B_{12M}^{(3)}(1, 1, 1)$	NYN	NYN	4	11	3	7	(18750, 31342)
103	157	$D_{12}^{(7)}(1, -1, -1, 1, -1, -1, 1)$	YYY	YYY	960	27	7	6	(33270, 106080)
104	158, 280	$D_{12}^{(7)}(-1, -1, 1, \mathbf{i}, -1, -1, -1)$	NYN	NYN	160	17	7	5	(29510, 85680)
105	159, 225	$D_{12}^{(7)}(1, -\mathbf{i}, -1, \mathbf{i}, -1, 1, -1)$	NYN	NYN	64	19	7	6	(27550, 69280)
106	160, 284	$D_{12}^{(7)}(1, -1, -1, 1, -1, -\mathbf{i}, -1)$	NYN	NYN	32	19	7	5	(29078, 70144)
107	161, 183	$D_{12}^{(7)}(1, 1, 1, 1, -1, 1, -1)$	NYN	NYN	192	25	7	6	(32622, 79488)
108	162, 252	$D_{12}^{(7)}(-1, -1, -\mathbf{i}, 1, -\mathbf{i}, -1, -1)$	NYN	NYN	64	19	7	6	(27558, 74208)
109	163, 216	$D_{12}^{(7)}(\mathbf{i}, \mathbf{i}, 1, 1, 1, -\mathbf{i}, -1)$	NYN	NYN	16	19	7	6	(27442, 66528)
110	164, 286	$D_{12}^{(7)}(\mathbf{i}, \mathbf{i}, 1, \mathbf{i}, \mathbf{i}, 1, -1)$	NYN	NYN	32	17	7	5	(28982, 67952)

ACT class	Equiv. classes	Family, coordinates	ACT	HBS	Auto. order	Defect	Orbit	$\mathbb{Z}_4$ rank	Invariant ( $4 \times 4, 5 \times 5$ )
111	165, 218	$D_{12}^{(7)}(-1, -i, 1, -i, i, i)$	NYN	NYN	192	25	7	6	(32358, 73632)
112	166, 219	$D_{12}^{(7)}(i, i, i, 1, i, i, -i)$	NYN	NYN	16	17	7	6	(27406, 66632)
113	167, 303	$D_{12}^{(7)}(1, 1, -1, -i, i, i, -1)$	NYN	NYN	32	13	7	5	(27070, 67008)
114	168, 304	$D_{12}^{(7)}(i, -i, i, 1, i, 1, -1)$	NYN	NYN	48	13	7	5	(26946, 64392)
115	169, 307	$D_{12}^{(7)}(-1, i, -i, -1, i, 1, -1)$	NYN	NYN	96	13	7	5	(26958, 70512)
116	170, 196	$X_{12}^{(7)}(-1, -1, i, -1, 1, i, i)$	NYN	NYN	64	19	7	6	(26870, 50752)
117	171, 186	$D_{12}^{(7)}(i, i, -i, -1, -i, i, 1)$	NYN	NYN	64	19	7	6	(27646, 76160)
118	172, 197	$X_{12}^{(7)}(1, -i, i, 1, -1, i, -1)$	NYN	NNN	16	15	7	5	(27314, 51064)
119	173, 212	$B_{12M}^{(3)}(1, i, i)$	NYN	NNN	16	13	3	5	(27056, 43288)
120	175, 208, 267, 274	$X_{12}^{(7)}(1, -1, -1, 1, i, 1, i)$	NNN	NNN	32	9	7	5	(25826, 42248)
121	176, 253	$X_{12}^{(7)}(i, i, -i, 1, i, -1, i)$	NYN	NYN	64	17	7	6	(27142, 47472)
122	178, 269	$X_{12}^{(7)}(1, i, -1, i, -1, -i, -1)$	NYN	NYN	32	19	7	5	(28254, 58488)
123	179, 224	$X_{12}^{(7)}(i, i, -i, 1, -i, 1, i)$	NYN	NYN	64	17	7	6	(26350, 45024)
124	181, 266	$X_{12}^{(7)}(i, i, -i, i, -1, -1, i)$	NYN	NYN	32	17	7	5	(27678, 45536)
125	182	$X_{12}^{(7)}(1, -1, -1, -i, -1, -i, -1)$	YYY	YYY	192	27	7	6	(32406, 57120)
126	184, 199	$X_{12}^{(7)}(1, 1, -i, -1, -1, -i, i)$	NYN	NYN	64	19	7	6	(26814, 47424)
127	185, 190	$D_{12}^{(7)}(i, 1, -1, 1, -1, -1, -1)$	NYN	NYN	64	19	7	6	(27494, 71648)
128	187, 282	$D_{12}^{(7)}(i, -i, 1, -i, 1, -i, 1)$	NYN	NYN	160	17	7	5	(29030, 74720)
129	188, 300	$D_{12}^{(7)}(-1, 1, 1, -i, -1, -1, i)$	NYN	NYN	32	17	7	5	(27078, 70272)
130	189, 306	$D_{12}^{(7)}(i, -1, -i, i, i, 1, 1)$	NYN	NYN	96	21	7	5	(27246, 77088)
131	191, 302	$D_{12}^{(7)}(1, -i, -i, 1, i, -1, -1)$	NYN	NYN	48	21	7	5	(26814, 66000)
132	192, 193	$B_{12B}^{(5)}(i, -i, -1, -1, 1)$	YNN	YYN	16	13	5	8	(20454, 40556)
133	195, 305	$X_{12}^{(7)}(1, i, -1, -1, 1, i, -1)$	NYN	NYN	32	17	7	5	(26090, 51248)
134	198, 301	$X_{12}^{(7)}(1, 1, -i, -1, -1, i, 1)$	NYN	NYN	32	19	7	5	(26006, 46256)
135	200, 217	$X_{12}^{(7)}(-1, i, -i, -1, -1, 1, -i)$	NYN	NNN	16	13	7	5	(27126, 44376)
136	201, 285	$X_{12}^{(7)}(i, -i, 1, 1, 1, -i, i)$	NYN	NNN	32	11	7	5	(27390, 40048)
137	202, 281	$X_{12}^{(7)}(-1, i, i, -i, -i, -1, -i)$	NYN	NNN	32	11	7	5	(27614, 45968)
138	203, 258	$D_{12}^{(7)}(i, -i, 1, -i, -i, i, 1)$	NYN	NYN	960	25	7	6	(31950, 76800)
139	204, 257	$X_{12}^{(7)}(1, -1, i, -i, -i, -1, i)$	NYN	NNN	192	19	7	6	(31302, 45600)
140	205, 242	$X_{12}^{(7)}(-1, -1, -i, i, -i, -1, i)$	NYN	NNN	192	19	7	6	(30990, 41088)
141	206, 262	$B_{12B}^{(5)}(1, 1, -1, -1, -1)$	NYN	NYN	64	15	5	6	(20318, 42192)
142	207, 259	$B_{12B}^{(5)}(-1, -1, -1, -1, -1)$	NYN	NYN	64	17	5	6	(20062, 38544)
143	210, 261	$B_{12B}^{(5)}(i, i, 1, i, 1)$	NYN	NYN	64	15	5	6	(20606, 37936)
144	211, 260	$B_{12B}^{(5)}(1, -1, -1, -1, 1)$	NYN	NYN	64	17	5	6	(20654, 36592)
145	213, 308	$B_{12M}^{(3)}(i, 1, -1)$	NYN	NYN	48	19	3	5	(25542, 42240)
146	214, 283	$B_{12N}^{(6)}(-1, 1, i, i, 1, -1)$	NYN	NNN	40	9	6	5	(27260, 37940)
147	215, 256	$B_{12H}^{(6)}(-1, -i, 1, 1, -1, 1)$	NYN	NNN	240	17	6	5	(31140, 37080)
148	220, 223	$B_{12I}^{(4)}(i, 1, 1, i)$	NYN	NYN	16	15	4	8	(19382, 29480)
149	221, 311	$B_{12O}^{(3)}(1, 1, 1)$	NYN	NNN	16	11	3	7	(18036, 22832)
150	222	$B_{12I}^{(4)}(i, 1, 1, 1)$	YYY	YYY	16	19	4	8	(19982, 33920)
151	226, 227, 287, 289	$B_{12P}^{(4)}(1, 1, 1, 1)$	NNN	NYN	16	10	4	6	(21018, 41104)
152	235, 290	$B_{12P}^{(4)}(1, 1, 1, -1)$	NYN	NYN	8	15	4	6	(21182, 40566)
153	236, 313	$B_{12B}^{(5)}(-1, -1, i, 1, i)$	NYN	NYN	8	11	5	7	(18600, 36956)
154	237, 248	$X_{12}^{(7)}(-1, i, -1, 1, 1, -1, i)$	NYN	NNN	48	15	7	5	(29142, 49512)
155	238, 246	$B_{12Q}^{(4)}(1, 1, 1, 1)$	NYN	NNN	16	11	4	5	(24692, 38312)
156	239, 245	$B_{12A}^{(4)}(i, 1, -i, 1)$	NYN	NNN	16	11	4	7	(19578, 31544)
157	240, 244	$B_{12A}^{(4)}(i, -1, -i, 1)$	NYN	NNN	16	11	4	7	(19794, 30180)
158	241, 298	$B_{12B}^{(5)}(i, -i, -i, 1, i)$	NYN	NYN	32	11	5	6	(20686, 34552)
159	247, 250	$B_{12Q}^{(4)}(1, 1, -1, 1)$	NNY	NNY	32	11	4	5	(24960, 45408)
160	249	$X_{12}^{(7)}(-1, i, 1, -1, 1, -1, i)$	YYY	NNY	48	15	7	5	(29364, 56832)
161	251, 296	$B_{12B}^{(5)}(-1, -1, 1, -1, i)$	NYN	NYN	32	13	5	6	(21222, 49720)
162	254, 297	$B_{12B}^{(5)}(-1, 1, i, i, i)$	NYN	NYN	32	13	5	6	(21806, 46744)
163	255, 299	$B_{12B}^{(5)}(i, i, -1, i, -i)$	NYN	NYN	32	11	5	6	(21334, 32856)
164	314, 317	$L_{12A}^{(1)}(1)$	NYN	NYN	8	7	1	9	(14130, 19680)
165	315, 319	$L_{12A}^{(1)}(-1)$	NYN	NNN	24	9	1	9	(14016, 17172)
166	316	$L_{12B}^{(1)}(i)$	YYY	NYN	8	11	1	9	(14284, 23940)
167	318	$L_{12B}^{(1)}(1)$	YYY	NNY	24	9	1	9	(13440, 8640)